HOMOTOPY GROUPS OF CONTACT 3-MANIFOLDS

by

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DEDICATION

To Norah Esty - Thank you for pushing me to leave my comfort zone.

There is something wonderful in wandering and exploring past previous boundaries. You did not give me a map, nor a pair of shoes.

You showed me that I have feet.

You are a great teacher, a fantastic mentor, and an amazing friend.

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[1] I forgive you for forgetting the sign.

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ABSTRACT

A contact 3-manifold (M, ξ) is an three-dimensional manifold endowed with a completely nonintegrable distribution. In studying such a space, standard homotopy groups, which are defined using continuous/smooth maps, are not useful as they are not sensitive to the distribution. To remedy this, we consider horizontal homotopy groups which are defined using horizontal maps, i.e., smooth maps that lie tangent to the distribution at every point. Due to the distribution being completely nonintegrable, horizontal maps into (M, ξ) have rank at most 1. This is used to show that the first horizontal homotopy group is uncountably generated and indicates that the higher horizontal homotopy groups are trivial.

We also consider Lipschitz homotopy groups which are defined using Lipschitz maps. We first endow (M, ξ) with a metric that is sensitive to the distribution, the Carnot-Carathéodory metric. With respect to this metric structure, the contact 3-manifold is purely 2-unrectifiable. This is used to show that the first Lipschitz homotopy group is uncountably generated and all higher Lipschitz homotopy groups are trivial. Furthermore, over the contact 3-manifold is a metric space, called the universal path space, that acts as a universal cover of the contact 3-manifold in that the universal path space is Lipschitz simply-connected and has a unique lifting property.

Homotopy groups, horizontal homotopy groups, and Lipschitz homotopy groups are all instances of homotopy groups of sheaves, which are defined.

INTRODUCTION

In this dissertation, we study the geometry of contact 3-manifolds via smooth and metric means. More specifically, we will define notions of homotopy groups that feel the contact structure before determining properties that these homotopy groups of contact 3-manifold must possess. The aim of this dissertation is to capture a sense in which a contact 3-manifold is an Eilenberg-MacLane $K(\pi, 1)$ space.

Contact manifolds, and in particular contact 3-manifolds, have inspired much interest in the last half century. In 1971, Martinet showed that any closed and orientable 3-manifold admits a contact structure, indicating a prevalence of contact 3-manifolds. Attempts to classify contact 3-manifolds were complicated by the work of Bennequin, who showed that $\mathbb{R}^3$ admits contact structures that are distinct from the standard contact structure, denoted $\mathbb{H}^1$. In 1989, Eliashberg proceeded to show that contact structures on 3-manifolds fall into one of two classes: tight and overtwisted. Inspecting these classes by means of Reeb vector fields and singular foliations have been the primary strategy towards better understanding contact 3-manifolds in the recent decades since. For modern accounts of the development of these approaches, see [9], [13], [14], or [23].

Rather than inspecting contact 3-manifolds via these standard techniques, we will proceed by probing the space with smooth maps that feel the contact structure. As the contact structure of a contact manifold is identified by a distribution, i.e., a sub-vector bundle of the tangent bundle, it is natural to consider smooth maps into a contact manifold that are tangent to this distribution. Such maps are referred to as horizontal.

In order to report the results of this probing, we define a notion of horizontal homotopy groups, denoted π_n^{H}. This definition first appears in the work of [6] as smooth horizontal homotopy groups. As is the case in algebraic topology, horizontal homotopy groups aim to capture various dimensional holes in a given space.

Standard homotopy groups will not be helpful in studying the contact structure of a contact manifold, as is seen in considering $\mathbb{R}^3$. Since $\mathbb{R}^3$ is contractible, all associated homotopy groups are trivial no matter the contact structure. But, as is shown in [6] for the standard contact structure $\mathbb{H}^1$ and as will be shown in this dissertation for any contact structure, the first horizontal homotopy group of $\mathbb{R}^3$ is non-trivial. Moreover, for any contact 3-manifold, the following will be shown:

Theorem 1.0.1. *For any contact 3-manifold, the first horizontal homotopy group is uncountably generated.*

As is the case with a smooth manifold (with the smooth maps into the smooth manifold), a contact manifold (with the horizontal maps into the contact manifold) forms a sheaf on the smooth site. This theorem then provides an immediate corollary:

Corollary 1.0.2. *As a sheaf, a contact 3-manifold is not representable, that is, there is no smooth manifold N such that smooth maps into the manifold N are equivalent to horizontal maps into the contact 3-manifold.*

The higher horizontal homotopy groups of a contact 3-manifold will be considered as well. Though we will not complete the calculation, we will provide a strong indication that these groups are trivial no matter the contact 3-manifold. This indication will be given by considering smooth maps from the n-sphere which are of rank at most 1, which is a trait all horizontal maps in to a contact 3-manifold will be shown to possess.

Theorem 1.0.3. *For $n > 1$, any smooth map of rank at most 1 from the n-sphere $\mathbb{S}^n$ into a manifold is smoothly null-homotopic.*

A benefit to considering horizontal homotopy groups is that they can be applied to any manifold endowed with a distribution, not just contact 3-manifolds. In particular, the work in this dissertation is applicable to Carnot manifolds with 2-dimensional bracket-generating distributions [27]. Another benefit is that results found in this dissertation are applicable to any contact 3-manifold, as opposed to many results in contact topology where the contact manifold often has the extra assumption of being co-orientable.

Alas, as the higher horizontal homotopy groups of a contact 3-manifold will not be calculated, we are unable to say definitively and precisely that contact 3-manifolds are $K(\pi, 1)$ spaces. But, by adding extra metric structure to the space, we are able to complete the desired calculations by considering Lipschitz homotopy groups rather than horizontal homotopy groups.

Lipschitz homotopy groups, denoted π_n^{Lip} are introduced in [6] as a means of studying sub-Riemannian manifolds. Sub-Riemannian manifolds naturally arise in settings where there is a restriction of motion and a desire to determine a most efficient path between points. Examples of such settings include describing the motion of a robot arm, the movement of a car, or the orbital dynamics of a satellite.

A sub-Riemannian manifold captures a restriction of motion via a distribution, a sub-vector bundle of the tangent bundle, and has a smoothly-varying inner product on the distribution so that questions about efficiency can be contemplated. The inner product attached to a sub-Riemannian manifold allows for a path metric, the Carnot-Carathéodory metric, where the infimum is taken over paths that are allowable with respect to this restriction of motion, i.e., paths that are tangent to the distribution. The Chow-Rashevskii theorem, a foundational result of the field, guarantees that any

pair of points in the space can be joined by an allowable path and thus the metric is well-defined.

The shortest path between two points with respect to this metric structure need not be the 'obvious' shortest path with respect to one's intuition. For example, consider the space of configurations of a car in a road. The configuration is determined by the direction the wheels are facing and the point in the parking lot over which the center of mass of the car lies. Consider two points in this space where, for each point, the wheels of the car are facing straight ahead but the center of mass of the second point is just to the right of the first point. The 'obvious' shortest path would be to push the car to the right. But, the obvious restriction of motion given by how a car moves means that this path is not allowable. One must do a parallel parking maneuver in order to produce a path from the first point to the second. As such, with this example and sub-Riemannian manifolds in general, we do not expect the geodesics or the metrics to act in tame ways.

An example of a sub-Riemannian manifold that has interesting sub-Riemannian structure is the first Heisenberg group. As a manifold, this space is three dimensional, but is four-dimensional as a metric space (Example in Section 2.8 in [24]).

This space is denoted $\mathbb{H}^1$. That this notation agrees with the standard contact structure on $\mathbb{R}^3$ is no coincidence as this metric space in the contact 3-manifold with extra metric structure attached.

Thus, sub-Riemannian manifolds provide interesting examples of metric spaces, $\mathbb{H}^1$ being the simplest non-trivial example. So, the first Heisenberg group $\mathbb{H}^1$ is often studied via metric tools to better understand its structure. For example, Dejarnette et al., first introduced Lipschitz homotopy groups in order to study Sobolev mappings into $\mathbb{H}^1$ [6]. Since Lipschitz homotopy groups were introduced, they have been calculated for various Heisenberg groups in [6], [17], [19], [18], and [34]. In

this dissertation, we expand these calculations past Heisenberg groups to contact 3-manifolds.

Lipschitz homotopy groups are a less natural, but more fruitful, tool than horizontal homotopy groups. Horizontal homotopy groups are a natural tool for studying contact manifolds as these groups capture the result of probing the space with smooth maps without introducing further structure on the space. Alas, π_n^{H} can be difficult to calculate, as is illustrated in this dissertation when $n > 1$. Though introducing a compatible metric adds more structure to the space, and thus detracts from the pureness of the probing, the Lipschitz homotopy groups of a contact manifold are easier to access as 'smooth' can be too strict of a condition to work with. By weakening 'smooth' to 'Lipschitz' and π_n^{H} to π_n^{Lip}, some calculations become easier. See [6] and [34] for papers inspecting the differences between these homotopy groups.

For our purposes, we will be able to complete the calculations in the Lipschitz case that we had intended to complete in the horizontal case:

Theorem 1.0.4. *For a contact 3-manifold endowed with a sub-Riemannian structure, and thus a Carnot-Carathéodory metric d_{CC}^M, the following is known about its Lipschitz homotopy groups:*

$$\pi_n^{Lip}(M, d_{CC}^M) = \begin{cases} \textit{uncountabley generated} & \textit{if } n = 1 \\ 0 & \textit{if } n \geq 2. \end{cases}$$

Furthermore, for an open contacto-embedding

$$f : (M', \xi') \longrightarrow (M, \xi)$$

between based contact 3-manifolds, the homomorphism induced on the first Lipschitz

homotopy groups

$$f_{\#} : \pi_1^{Lip}(M', d_{CC}^{M'}) \longrightarrow \pi_1^{Lip}(M, d_{CC}^{M})$$

is injective.

The argument for proving Theorem 1.0.1 will closely follow the logic outlined in Proposition 4.7 and Theorem 4.11 (2) in [6]. In fact, the argument proving Theorem 1.0.4 for the case where $n = 1$ is nearly identical to the arguments for Proposition 4.7 and Theorem 4.11 (2) in [6] and Theorem 1.0.4 is an extension of Theorem 4.11 (2), in the case where $n = 1$, from the first Heisenberg group to any contact 3-manifold.

Along the way in proving Theorem 1.0.4, we also extend Theorem 7.2 of [1] from the first Heisenberg group to any contact 3-manifold by showing that contact 3-manifolds (with a sub-Riemannian structure) are purely 2-unrectifiable. With this fact in hand, Theorem 1.0.4 for $n > 1$ will be immediate via Theorem 5 of [34].

Both Theorem 1.0.1 and Theorem 1.0.4 rely heavily on a fundamental theorem of contact topology, the Theorem of Darboux. As we proceed, we will spell out these connections.

In chapter 2, we will provide background on topics in differential and metric geometry that will be important in proceeding. In particular, contact manifolds, the object of primary interest to this dissertation, will be defined. Of note, a key observation about contact 3-manifolds will be proved which will state that smooth maps into a contact 3-manifold which respect the contact structure, i.e., horizontal maps, can have rank at most 1. This fact will be used significantly in chapters 4 and 5. We will also discuss sub-Riemannian geometry and a metric structure that we will put on any contact manifold, called the Carnot-Carathéodory metric.

In chapter 3, we will define homotopy groups of pointed sheaves on the smooth

site. The definition of homotopy groups of pointed sheaves is a generalization of homotopy groups of pointed topological spaces. That such a generalization exists is a folklore result that will be made rigorous. We will begin with a review of sheaves, natural transformations, and the Yoneda lemma before proceeding into defining homotopy groups of a sheaf as a set. Via Yoneda lemma, this definition will be obviously analogous to the standard definition of homotopy groups on a manifold. We will then proceed to use the sheaf condition to define a group operation akin to concatenation. With this notion of homotopy groups in hand, we will define a pair of sheaves on the smooth site. The first will depend on the contact structure of a contact manifold and the homotopy groups of this sheaf will be the horizontal homotopy groups of the contact manifold. The second will depend on a metric structure that we placed on contact manifolds and the homotopy groups of this sheaf will be the Lipschitz homotopy groups of the contact manifold.

In chapter 4, we will show that the first horizontal homotopy group of any contact 3-manifold is uncountably generated. First, we will consider two horizontal embeddings of $\mathbb{S}^1$ into a contact 3-manifold such that the images of the embeddings are not equal. We will show that any smooth homotopy between such embeddings must have rank 2 somewhere. The key observation of chapter 2 will then imply that no horizontal homotopy exists between distinct horizontal embeddings of $\mathbb{S}^1$ into a contact 3-manifold. It will then be shown that there are uncountably many such horizontal embeddings, and thus uncountably many elements of the first horizontal homotopy group, and that groups with uncountable cardinality are uncountably generated. It will follow that the sheaf associated to the contact structure of a contact 3-manifold is not representable.

In chapter 5, we indicate an approach for showing that the nth horizontal homotopy groups of a contact 3-manifold is trivial for $n > 1$. Due to the key

observation of chapter 2, all horizontal maps from $\mathbb{S}^n$ are rank at most 1 and thus, this chapter focuses on smooth maps from the n-sphere which have rank 1. From such a smooth map, we will show that there is an equivalence relation on the n-sphere such that the map is constant on each equivalence class and the associated quotient space is a tree. This will follow from the Jordan-Brouwer separation theorem as many of these equivalence classes will be codimension 1 compact submanifolds of $\mathbb{S}^n$. Thus, the smooth map will continuously factor through a contractible space and thus be continuously, and therefore smoothly, null-homotopic. Alas, this will not be enough to guarantee that horizontal maps of the the n-sphere are horizontally null-homotopic.

In chapter 6, we begin our study of the sub-Riemannian geometry of contact 3-manifolds and show that the nth Lipschitz homotopy group of a contact 3-manifold is trivial for $n > 1$. This will be an immediate consequence to showing that contact 3-manifolds are purely 2-unrectifiable, a metric condition akin to the key observation of chapter 2. Essentially, this condition says that no Lipschitz map out of 2-dimensional Euclidean space into a contact 3-manifold can sweep out area in the target space. It is known that the contact structure of contact 3-manifolds is locally modeled after the first Heisenberg group $\mathbb{H}^1$ (the theorem of Darboux) and that $\mathbb{H}^1$ is purely 2-unrectifiable [1]. By inspecting the interplay between distributional embeddings and Carnot-Carathéodory metrics, we will show that there is a BiLipschitz version of the Theorem of Darboux which will state that the Carnot-Carathéodory metric structure of contact 3-manifolds is locally modeled after the metric structure of $\mathbb{H}^1$.

In chapter 7, we will show that the first Lipschitz homotopy group of a contact 3-manifold is uncountably generated. The argument verifying this statement will be very close to the one used in chapter 4 to show the horizontal case. We will again consider two horizontal, and thus Lipschitz, embeddings of $\mathbb{S}^1$ into a contact 3-manifold (with a Carnot-Carathéodory metric) such that the images of

the embeddings are not equal. It will be shown that the 2-dimensional Hausdorff measure of the image of a Lipschitz homotopy between the embeddings is positive. But, since contact 3-manifolds are purely 2-unrectifiable, no such Lipschitz homotopy exists and every pair of distinct horizontal embeddings of $\mathbb{S}^1$ into a contact 3-manifold yields a distinct element of the first Lipschitz homotopy group.

BACKGROUND

Throughout this dissertation, the term "manifold" will refer to "smooth manifold." We will specifically note when a manifold potentially has boundary. The tangent bundle of a manifold M will be denoted TM.

Manifolds with distributions

The main purpose of this dissertation is to investigate contact 3-manifolds, which are 3-manifolds endowed with a distribution satisfying a complete non-integrability condition. Distributions will also be an essential structure for understanding horizontal maps into contact 3-manifolds as a contact structure is an example of a manifold endowed with a distribution. Thus, we define what is meant by a distribution now.

Definition 2.0.1. For a manifold M, a *co-dimension k distribution ξ* on the manifold M is a co-dimension k vector sub-bundle $\xi \subset TM$ of the tangent bundle TM. This will be denoted as a pair (M, ξ). We will refer to this pair as *a manifold with a distribution* or *a manifold endowed with a distribution.* At a point $p \in M$, the vector subspace of the tangent space T_pM assigned by the distribution will be denoted ξ_p.

Example 2.0.2. Endow the manifold $\mathbb{R}^n$ with the co-dimension 1 distribution $T\mathbb{R}^{n-1} \times \mathbb{R}$. At each point $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$, the distribution assigns the co-dimension 1 vector space $T_{(x_1,\ldots,x_{n-1})}\mathbb{R}^{n-1} \times \{0\} \subset T_{(x_1,\ldots,x_n)}\mathbb{R}^n$ of the tangent space at $(x_1, \ldots, x_n)$, that is, the horizontal hyperplanes of the tangent space.

As will be explained in the next section, distributions capture an idea of a restriction of instantaneous motion at each point of a manifold. Indeed, the tangent space at a point can be thought of as all possible velocities that can be achieved via

smooth paths at that point. This is inherent in the definition of the tangent space as an element of the tangent space is an equivalence class of paths traveling through the point, all of which have the same velocity. As such, a distribution at a point gives restrictions on the possible tangent vectors, i.e., velocities, that can be attained.

Smooth, Horizontal, and Distributional maps

Smooth maps are the first tool employed in this dissertation for probing contact manifolds. In particular, a certain sort of smooth map, called a horizontal map, is sensitive to a distribution and is used in the definition of π_n^{H}. Later on in this chapter, horizontal paths will also be used to define a metric, called the Carnot-Carathéodory metric, on a larger class of spaces than just contact manifolds, called sub-Riemannian manifolds.

Horizontal maps will be used more generally than just probing contact manifolds as they are defined for any manifold endowed with a distribution. As such, this section focuses on manifolds endowed with a distribution. Further, we define a stronger notion of mapping between manifolds endowed with distributions, called distributional mappings. These maps send the distribution of the domain into the distribution of the target.

Due to the importance of smooth maps to this dissertation, we recall several well-known definitions and theorems in this section, along with formally introducing horizontal and distributional maps.

Notation 2.0.3. The *derivative* (or *pushforward*) of a smooth map $f : N \longrightarrow M$ is the smooth vector bundle map

$$\begin{tikzcd}
TN \arrow[r, "Df"] \arrow[d] & TM \arrow[d] \\
N \arrow[r, "f"] & M,
\end{tikzcd}$$

where the map Df restricted to the fiber over a point $p \in N$ is the linear map denoted

$$D_p f : T_p N \longrightarrow T_{f(p)} M.$$

Theorem 2.0.4. *(Chain Rule) For smooth maps $f : M' \longrightarrow M$ and $g : N \longrightarrow M'$, the derivative of the composition $f \circ g$ agrees with the compositions of the individual derivatives of the maps f and g:*

$$\begin{tikzcd}
& TM' \arrow[r, "Df"] \arrow[dd] & TM \arrow[dd] \\
TN \arrow[ur, "Dg"] \arrow[urr, bend right, "D(f\circ g)"'] \arrow[dd] & & \\
& M' \arrow[r, "f"] & M. \\
N \arrow[ur, "g"] \arrow[urr, bend right, "f\circ g"'] & &
\end{tikzcd}$$

Definition 2.0.5. Let $f : N \longrightarrow M$ be a smooth map between manifolds N and M. For any point $p \in N$, the *rank* of f at p is the rank of the linear map

$$D_p f : T_p N \longrightarrow T_{f(p)} M,$$

i.e., the dimension of the image of $D_p f$.

The map f is of rank k on a subset A if, for all $p \in A$, the rank of f at p is k. Similarly, the map f is of rank less than k on a subset A if, for all $p \in A$, the rank of f at p less than k.

With some important notions of smooth maps revisited, we define horizontal maps. The definition of horizontal maps takes advantage of smooth maps naturally yielding maps between tangent spaces. Thus, one can ask to only look at maps whose derivative factors through some given distribution.

Definition 2.0.6. Let (M, ξ) be a manifold endowed with a distribution. A map is *horizontal* if it is tangent to the distribution. That is, for a smooth map $f : N \longrightarrow M$, the derivative map factors through the distribution ξ:

$$\begin{tikzcd}
 & \xi \arrow[d, hook] \\
TN \arrow[ur, dashed, "\exists"] \arrow[r, "Df"] \arrow[d] & TM \arrow[d] \\
N \arrow[r, "f"] & M.
\end{tikzcd}$$

The horizontal map f is denoted $f : N \longrightarrow (M, \xi)$. Denote the set of all horizontal maps from N into (M, ξ) by $\mathsf{Map}^{\mathsf{H}}\big(N, (M, \xi)\big)$.

When N is a closed interval, the map f is called a *horizontal path*. The collection of horizontal paths into a manifold endowed with a distribution (M, ξ) is denoted $\mathsf{Path}^{\mathsf{Horiz}}(M, \xi)$.

Here, we have chosen to define horizontal paths to be smooth, as is done by Montgomery in [24]. Alternatively, we could have asked that horizontal paths be absolutely continuous with respect to some chosen Riemannian metric on M, as is done in [6]. Absolutely continuous maps are differentiable almost everywhere, so it is reasonable to ask that, when they exist, that the derivatives satisfy a certain property. In the two referenced works, the choice is made in defining the Carnot-Carathéodory metric for a sub-Riemannian manifold, which we will define later this chapter.

Example 2.0.7. For the manifold endowed with a distribution $(\mathbb{R}^n, T\mathbb{R}^{n-1} \times \mathbb{R})$ described in Example 2.0.2, each path ι_i into $\mathbb{R}^n$ given by $\iota_i(t) = (0, \dots, t, \dots, 0)$ where t appears in the ith coordinate, is horizontal for $i \neq n$. Indeed, for all t in the domain and each $i = 1, \dots, n-1$,

$$\iota_i'(t) = (0, \dots, 1, \dots, 0) \in T_{(0,\dots,t,\dots,0)}\mathbb{R}^{n-1} \times \{0\}$$

and the linear map $D_t\iota_i$ maps into the distribution over the point $(0, \dots, t, \dots, 0)$.

In the event that $i = n$, the derivative for all time t is $\iota_n'(t) = (0, \dots, 0, 1)$ which is not in the distribution. Thus, the map ι_n is not horizontal with respect to this distribution.

For a manifold endowed with a distribution (M, ξ), horizontal paths capture a global sense in which a distribution enacts a restriction of motion within M. A tangent vector at a point $p \in M$ can be thought of as a velocity vector for some unspecified path through p. As such, a tangent vector gives instructions for first-order movement from p. A distribution then restricts possible first-order movement at any point in M. A horizontal path is then seen as movement within the space that adheres to this restriction of motion for all time as each velocity vector lies tangent to the distribution. The horizontal path is enacting first-order allowable movement at all times.

Due to the Chain Rule, post-composing any smooth map by a horizontal map yields another horizontal map.

Lemma 2.0.8. *Let $f : N \longrightarrow (M, \xi)$ be a horizontal map from a manifold N into a manifold with a distribution (M, ξ). Let $g : N' \longrightarrow N$ be a smooth map from a manifold N'. Then*

$$f \circ g : N' \longrightarrow (M, \xi)$$

is a horizontal map into (M, ξ).

Proof. This follows immediately from the chain rule. The diagram succinctly describes the argument:

$$\begin{tikzcd}
& & \xi \arrow[d, hook] \\
& TN \arrow[ur] \arrow[r, "Df"] \arrow[dd] & TM \arrow[dd] \\
TN' \arrow[ur, "Dg"] \arrow[uurr, dashed, "\exists"'] \arrow[urr, bend right, "D(f\circ g)"'] \arrow[dd] & & \\
& N \arrow[r, "f"] & M. \\
N' \arrow[ur, "g"] \arrow[urr, bend right, "f\circ g"'] & &
\end{tikzcd}$$

□

Further, one can consider maps whose derivatives do not send the entire tangent space into a specified distribution over the target, but rather a distribution on the domain into a distribution over the target. This notion is captured with distributional maps.

Definition 2.0.9. Let (M, ξ) and (M', ξ') be manifolds endowed with a distribution. A smooth map $f : M' \longrightarrow M$ is a *distributional map*, denoted

$$f : (M', \xi') \longrightarrow (M, \xi),$$

if its derivative maps one distribution into the other:

$$\begin{tikzcd}
\xi' \arrow[r, dashed, "\exists"] \arrow[d, hook] & \xi \arrow[d, hook] \\
TM' \arrow[r, "Df"] \arrow[d] & TM \arrow[d] \\
M' \arrow[r, "f"] & M.
\end{tikzcd}$$

If, in addition to being distributional, f is a smooth embedding, f is referred to as a *distributional embedding.* If, in addition to being distributional, f is a diffeomorphism (and f^{-1} is distributional), f is referred to as a *distributional diffeomorphism.*

Remark 2.0.10. In the literature, in the case where a distributional diffeomorphism is between contact manifolds, it is referred to as a contactomorphism.

Observation 2.0.11. Let (M, ξ) be a manifold endowed with a distribution and $f : N \longrightarrow M$ be a smooth map. If the derivative $D_p f$ vanishes at a point $p \in N$, then the map f is distributional at p, that is, for any tangent vector $v \in T_p N$,

$$D_p f(v) \in \xi_{f(p)}.$$

Indeed, this is the case as $D_p f(v) = 0$ and the zero vector is in the vector space $\xi_{f(p)}$.

Horizontal maps are more restrictive than distributional maps. Indeed, any horizontal map is a distributional map when the tangent bundle itself is taken as the distribution on the domain. But it is unlikely that a distributional map is horizontal. For example, on any manifold M with a distribution $\xi \neq TM$ that is not the entire tangent bundle, the identity map $\mathbb{1}_M : (M, \xi) \to (M, \xi)$ is distributional but not horizontal.

The main reason to consider distributional maps for the purposes of this dissertation is due to their ability to capture when two manifolds with distributions are the same, at least locally. Later on in this chapter, Darboux's Theorem will

be introduced, which guarantees that for each point in a contact 3-manifold, there is a distributional embedding of the 3-dimensional Heisenberg group $\mathbb{H}^1$ onto a neighborhood of the point. This will capture that contact 3-manifolds are modeled locally by $\mathbb{H}^1$.

As is the case when smooth maps are post-composed by horizontal maps, post-composing a horizontal map by a distributional map yields a horizontal map, which is indicated in the following lemma.

Lemma 2.0.12. *Let $f : (M', \xi') \longrightarrow (M, \xi)$ be a distributional map between manifolds with distributions. Let $g : N \longrightarrow (M', \xi')$ be a horizontal map from a manifold N. Then, the composition*

$$f \circ g : N \longrightarrow (M, \xi)$$

is a horizontal map into (M, ξ).

Proof. This follows immediately from the chain rule. The diagram succinctly describes the argument:

$$\begin{tikzcd}
 & \xi' \arrow[r] \arrow[d, hook] & \xi \arrow[d, hook] \\
 & TM' \arrow[r, "Df"] \arrow[dd] & TM \arrow[dd] \\
TN \arrow[uur] \arrow[ur, "Dg"] \arrow[uurr, dashed, bend right, "\exists"'] \arrow[urr, bend right, "D(f\circ g)"'] \arrow[dd] & & \\
 & M' \arrow[r, "f"] & M. \\
N \arrow[ur, "g"] \arrow[urr, bend right, "f\circ g"'] & &
\end{tikzcd}$$

□

We note that the existence of distributional diffeomorphism yields a bijection between the sets of horizontal paths on the domain and target.

Proposition 2.0.13. *Let* $\psi : (M', \xi') \longrightarrow (M, \xi)$ *be a distributional diffeomorphisms between manifolds with distributions. Then* ψ *induces an isomorphism on the sets of horizontal paths via derivative:*

$$\psi_* : \mathsf{Path}^{\mathsf{Horiz}}(M', \xi') \xrightarrow{\cong} \mathsf{Path}^{\mathsf{Horiz}}(M, \xi)$$

$$\gamma \longmapsto \psi \circ \gamma.$$

Proof. To show that the map ψ_* is injective, take horizontal paths $\gamma, \gamma' \in \mathsf{Path}^{\mathsf{Horiz}}(M', \xi')$ such that their images under ψ_* are equal: $\psi \circ \gamma = \psi \circ \gamma'$. Since the map ψ is invertible, the inverse map $\psi^{-1} : (M, \xi) \longrightarrow (M', \xi')$ exists and is a distributional map. Thus, the horizontal paths γ and γ' must agree:

$$\gamma = \psi^{-1} \circ (\psi \circ \gamma) = \psi^{-1} \circ (\psi \circ \gamma') = \gamma',$$

and ψ_* is injective.

For surjectivity, let $\alpha \in \mathsf{Path}^{\mathsf{Horiz}}(M, \xi)$ be a horizontal path in (M, ξ). Since the map ψ^{-1} is distributional, by Lemma 2.0.12, the composition $\psi^{-1} \circ \alpha$ is a horizontal path in (M', ξ') such that its image under ψ_* is the horizontal path α. Thus, the map ψ_* is surjective and a bijection. □

Vector fields and Lie brackets

Vector fields will be a useful tool for understanding distributions and, more so, their Lie brackets will be essential for interpreting their global structure. We recall these concepts now.

Definition 2.0.14. For a fiber bundle $\pi : E \longrightarrow B$, a *section* is a smooth map $X : B \longrightarrow E$ such that the following diagram commutes:

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ B \arrow[ur, "X"] \arrow[r, "\mathbb{1}_B"'] & B. \end{tikzcd}$$

Denote the collection of all sections of the fiber bundle $\pi : E \longrightarrow B$ by $\Gamma(E, B)$.

Definition 2.0.15. For a manifold M, a *vector field* is a section of the tangent bundle $TM \longrightarrow M$. The collection of all vector fields on M is denoted $\mathfrak{X}(M)$.

Definition 2.0.16. For a manifold endowed with a distribution (M, ξ), a vector field $X \in \mathfrak{X}(M)$ is *tangent to the distribution* if the vector field factors through the distribution:

$$\begin{tikzcd} \xi \arrow[r, hook] & TM \arrow[d] \\ B \arrow[u, dashed, "\exists"] \arrow[ur, "X"] \arrow[r, "\mathbb{1}_B"'] & B. \end{tikzcd}$$

Vector fields on a manifold act as derivations. In particular, a vector field $X \in \mathfrak{X}(M)$ and a smooth, real-valued map $f : M \longrightarrow \mathbb{R}$ determines another smooth, real-valued map $Xf : M \longrightarrow \mathbb{R}$ given by

$$Xf(p) = X_p f = D_p f(X_p)$$

for any point p in M. Thus, given another vector field $Y \in \mathfrak{X}(M)$, we can compose these vector fields and have the result act on a smooth, real-valued function to yield another smooth, real-valued function: $Y(Xf)$. The composition YX is known to not be a derivation, but the following construction, called the Lie bracket, is a derivation. See Chapter 14 in [32] for proofs and further discussion.

Definition 2.0.17. For a manifold M and vector fields $X, Y \in \mathfrak{X}(M)$ the *Lie bracket* of X and Y, denoted $[X, Y]$, is the vector field on M defined by

$$[X, Y](f) := X(Yf) - Y(Xf)$$

for any smooth, real-valued function f on M.

For a distribution ξ on a manifold M, we can consider iterating the Lie brackets of vector fields tangent to the distribution. For a point $p \in M$, consider the subspace of T_pM given by

$$\operatorname{span}\{[X_1, [X_2, [\ldots, [X_{j-1}, X_j]\ldots](p) \;:\; X_i \in \Gamma(\xi, M),\; j \in \mathbb{N}\} \subset T_pM.$$

This subspace collects all possible tangent vectors in T_pM that can be attained by iterating the Lie bracket of vectors tangent to the distribution ξ.

Definition 2.0.18. For a manifold endowed with a distribution (M, ξ), the distribution ξ is *bracket-generating* if this iteration process recovers the entire tangent space for each point $p \in M$, that is,

$$\operatorname{span}\{[X_1, [X_2, [\ldots, [X_{j-1}, X_j]\ldots](p) \;:\; X_i \in \Gamma(\xi, M),\; j \in \mathbb{N}\} = T_pM.$$

Integrable distributions and foliations

Vector fields and Lie brackets are a useful tool in inspecting global properties of distributions. A distribution can be described locally via vector fields that lie tangent to the distribution. Further, the Lie bracket of vectors fields that lie tangent to the distribution can yield useful information about the distribution. In particular, consider when a distribution is *closed under Lie brackets*.

Definition 2.0.19. For a manifold endowed with a distribution (M, ξ), the distribution is *closed under Lie brackets*, or *involutive*, if, for any vectors fields $X, Y \in \mathfrak{X}(U)$ defined on any open set $U \subset M$ such that the vector fields are tangent to the distribution ξ, the Lie bracket yields a vector field $[X, Y]$ that is also tangent to the distribution.

Observation 2.0.20. A distribution that is closed under Lie brackets and is not the entire tangent bundle is as far away as possible from being bracket-generating.

Example 2.0.21. For any manifold M, the tangent bundle TM is a distribution. Since the Lie bracket of any pair of local vector fields on M is defined, the resulting vector field is tangent to TM. Therefore, TM is closed under Lie brackets.

Example 2.0.22. Returning to Example 2.0.2, the manifold endowed with a distribution $(\mathbb{R}^n, T\mathbb{R}^{n-1} \times \mathbb{R})$ is closed under Lie brackets. Indeed, since $[-,-]$ is bilinear, we only need to consider the vector fields

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}$$

scaled by smooth, real-valued functions on $\mathbb{R}^n$. Since such vector fields form a basis for all vector fields belonging to the distribution, it is enough to verify that the Lie bracket of these vector fields belong to $T\mathbb{R}^{n-1} \times \mathbb{R}$.. Let $f_i, f_j : \mathbb{R}^n \to \mathbb{R}$ be smooth functions and consider the vector fields $f_i \frac{\partial}{\partial x_i}$ and $f_j \frac{\partial}{\partial x_j}$ for indices $i, j \in \{1, \ldots n-1\}$. By standard differentiation techniques, there is an equality of vector fields:

$$\left[f_i \frac{\partial}{\partial x_i}, f_j \frac{\partial}{\partial x_j} \right] = \left(f_i \frac{\partial f_j}{\partial x_i} \right) \frac{\partial}{\partial x_j} - \left(f_j \frac{\partial f_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

Thus, the Lie bracket of $f_i \frac{\partial}{\partial x_i}$ and $f_j \frac{\partial}{\partial x_j}$ is contained in the distribution $T\mathbb{R}^{n-1} \times \mathbb{R}$.

When a distribution is closed under Lie brackets, the Frobenius integrability theorem will yield a global result, there is a *foliation* associated to the distribution.

Definition 2.0.23. Let M be an n-dimensional manifold. A *co-dimension* $n-k$ *foliation* of M is a collection of smooth immersions

$$\iota_\alpha : L_\alpha \longrightarrow M$$

with domains L_α which are k-dimensional connected manifolds, indexed by $\alpha \in J$, such that the collection of images of the immersions $\{\iota_\alpha(L_\alpha)\}_{\alpha \in J}$ is a partition of M and, for all points $p \in M$, there exists a neighborhood U and a local parameter $x : U \to \mathbb{R}^n$ such that

- $x(U) = (-\varepsilon, \varepsilon)^n$ for some $\varepsilon > 0$,
- $x(p) = (0, \ldots, 0)$,
- For each index $\alpha \in J$ and for each connected component of $\iota_\alpha(L_\alpha) \cap U$, there exists values $|a^i| < \varepsilon$ for $i = k+1, \ldots, n$, such that the connected component is equal to

  $$\{p' \in U \ : \ x^{k+1}(p') = a^{k+1}, \ldots, x^n(p') = a^n\},$$

  and
- For the immersed submanifold $\iota(L)$ containing p, the component of $\iota(L) \cap U$ containing p is equal to

  $$\{p' \in U : x^{k+1}(p') = \ldots = x^n(p') = 0\}.$$

Such a neighborhood U is referred to as a *foliated neighborhood* or a *foliational parametrization*. Each immersed submanifold $\iota_\alpha(L_\alpha)$ is referred to as a *leaf* of the

foliation.

Convention 2.0.24. Often, the immersion associated to a leaf $\iota_\alpha(L_\alpha)$ will be suppressed and we will refer to the leaf and the domain of the immersion as L_α except when this convention brings unnecessary confusion.

Example 2.0.25. A foliation of $\mathbb{R}^n$ is given by the collection of embedded submanifolds $\{\mathbb{R}^{n-1} \times \{t\}\}_{t\in\mathbb{R}}$. This is the standard co-dimension 1 foliation of $\mathbb{R}^n$. Note that, for each point $(x_1, \ldots, x_n) \in \mathbb{R}^n$, the point is in the leaf $\mathbb{R}^{n-1} \times \{x_n\}$. Also, the tangent space of this leaf as a submanifold of $\mathbb{R}^n$ agrees with the distribution described in Example 2.0.2.

Given a co-dimension $n-k$ foliation of a manifold M, for a point $p \in M$ and a the leaf L_α containing p, since immersions are locally embeddings, the tangent space of L_α at p is identified as a vector subspace of the tangent space at p:

$$T_pL_\alpha \subset T_pM.$$

It follows that a foliation yields a co-dimension $n-k$ distribution on the manifold. It is natural to then ask which distributions yield foliations. We call such a distribution *integrable.*

Definition 2.0.26. Let (M, ξ) be a manifold endowed with a distribution. the distribution ξ is *integrable* if there exists a foliation $\{L_\alpha\}_{\alpha\in J}$ such that, for any point $p \in M$, the distribution at p agrees with the tangent space of the leaf containing the point at p:

$$T_pL_\alpha = \xi_p.$$

By Frobenius integrability theorem, a distribution is integrable exactly when it is closed under Lie brackets.

Theorem 2.0.27. *(Frobenius integrability theorem [12]) For a manifold endowed with a distribution* (M, ξ)*, the distribution* ξ *is integrable if and only if the distribution* ξ *is closed under Lie brackets.*

Remark 2.0.28. This is an example of a local-to-global result as we only need to understand a local notion of the distribution, how Lie brackets of vector fields tangent to the distribution, in order to make a global conclusion, the manifold is partitioned by leafs of a foliation associated to the distribution.

Contact manifolds

Distributions with this integrability condition are, in a way, the exact opposite of the distributions that we are interested in studying. Rather than studying distributions that are closed under Lie brackets, we will be concerned with distributions that are not closed under Lie brackets at any point in the manifold. Further, we will ask that these distributions recover the entire tangent space at each point via bracketing vector fields tangent to the distribution, that is, that the distribution is bracket-generating.

Such distributions are also referred to as completely non-integrable in contrast to the integrable distribution discussed in Frobenius integrability theorem. Indeed, that theorem guarantees that, if a distribution is not closed under Lie brackets at every point of the manifold, there is no submanifold whose tangent space agrees with the distribution on any neighborhood.

Definition 2.0.29. A *contact manifold* is a pair (M, ξ) consisting of an odd-dimensional connected manifold M, and a bracket-generating codimension 1 distribution ξ. Such a distribution ξ is referred to as a *contact distribution.*

The primary source of examples of contact manifolds are the $(2n+1)$-dimensional Heisenberg groups. These examples are defined and explored later on in this chapter.

Observation 2.0.30. The main concern of this dissertation is to study contact 3-manifolds. For such spaces, the bracket-generating condition simplifies to the following: for all points $p \in M$, there exists a neighborhood $p \in U$ and vectors fields tangent to the contact distribution $X, Y \in \Gamma(\xi, M)$ such that there is an equality of vector spaces:

$$T_pM \cong \xi_p \bigoplus \operatorname{span}([X,Y]_p).$$

Example 2.0.31 (Airport bag example)**.** Consider a two-wheeled airport bag in an infinite airport.[1] Fix the angle between the floor of the airport and the handle of the bag. The manifold $\mathbb{R}^2 \times \mathbb{S}^1$ describes the possible configurations of the airport bag, where $\mathbb{R}^2$ captures the center of mass of the airport bag over some point in the airport and $\mathbb{S}^1$ captures the direction that the airport bag is facing.

There is also a restriction of possible motion of the airport bag. It is possible to rotate the bag to change its direction and it is possible to roll the bag in the direction the wheels are facing. It is not possible, however, to skid the bag in the perpendicular direction to the direction the wheels are facing. The friction is too great to overcome. Infinitesimally, the restriction of motion yields a distribution on the manifold $\mathbb{R}^2 \times \mathbb{S}^1$. For local coordinates x, y, and θ on the manifold, define the distribution ξ by the assignment

$$\xi_{(x,y,\theta)} \coloneqq \operatorname{span}\left(\frac{\partial}{\partial\theta}, \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right).$$

We will now show that the distribution ξ is a contact distribution. Take a smooth function $f : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}$. Consider how the (local) vector fields which span ξ act on

[1] If it is more evocative, the reader is encouraged to replace the airport bag with a grocery cart from your local Park 'n Shop.

f as compositions of derivations:

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right)(f) &= \frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial f}{\partial x}+\sin\theta\frac{\partial f}{\partial y}\right)\\ &= -\sin\theta\frac{\partial f}{\partial x}+\cos\theta\frac{\partial f}{\partial y}+\cos\theta\frac{\partial^2 f}{\partial x\partial\theta}+\sin\theta\frac{\partial^2 f}{\partial y\partial\theta},\end{aligned}$$

and

$$\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right)\frac{\partial}{\partial\theta}(f) = \cos\theta\frac{\partial^2 f}{\partial x\partial\theta}+\sin\theta\frac{\partial^2 f}{\partial y\partial\theta}.$$

Thus, the Lie bracket of the spanning vector fields acting as a derivation on the real-valued map f is the following smooth function:

$$\left[\frac{\partial}{\partial\theta},\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right](f) = -\sin\theta\frac{\partial f}{\partial x}+\cos\theta\frac{\partial f}{\partial y}.$$

Thus, the Lie bracket of the spanning vector fields is the vector field

$$\left[\frac{\partial}{\partial\theta},\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right] = -\sin\theta\frac{\partial}{\partial x}+\cos\theta\frac{\partial}{\partial y}.$$

The Lie bracket does not belong to the distribution ξ. Note that the Lie bracket describes the direction perpendicular to the direction the airport bag is facing. Thus, the spanning vector fields of the distribution ξ and the Lie bracket of the spanning vector fields span the tangent bundle $T(\mathbb{R}^2\times\mathbb{S}^1)$. Therefore, the manifold endowed with a distribution $(\mathbb{R}^2\times\mathbb{S}^1,\xi)$ is a contact 3-manifold.

Horizontally path-connected

A distribution on a manifold can be interpreted as a first-order restriction of motion on the underlying space. As has been noted, horizontal paths are a means of

witnessing this restriction of motion on a global scale as a horizontal path has velocity at each time that is tangent to the distribution. It is natural to ask if movement between any two points in the underlying manifold is possible via horizontal paths, that is, if the space is *horizontally path-connected*.

Definition 2.0.32. A manifold endowed with a distribution (M,ξ) is *horizontally path-connected* if, for any points $p, q \in M$, there exists a horizontal path

$$\gamma : [a,b] \longrightarrow (M,\xi)$$

such that $\gamma(a) = p$ and $\gamma(b) = q$.

In the case that the manifold M is connected and the distribution is the tangent bundle $\xi = TM$, notions of smooth path and horizontal path agree. Thus, the pair (M, TM) is horizontally path-connected as the manifold M is path-connected. As is shown below, this is the only case where an integrable distribution yields a horizontally path-connected space.

Proposition 2.0.33. *Let (M,Δ) be an n-dimensional manifold with a k-dimensional integrable distribution. Take $\gamma : \mathbb{R} \longrightarrow (M,\Delta)$ to be a smooth map that lies tangent to the distribution. Denote $L \subset M$ to be the leaf of the foliation associated to Δ containing $f(0)$. Then, $\operatorname{Im}\gamma \subset L$.*

Proof. Since the distribution Δ is integrable, there is a k-dimensional foliation of M.

Fix a time $t \in \mathbb{R}$. The point $\gamma(t)$ has a foliated neighborhood (x, U). The parametrization x is such that $x(\gamma(t)) = (0,\ldots,0)$ and takes image in $(-\varepsilon,\varepsilon)^n$ for some $\varepsilon > 0$. Since the parametrization x brings components of leafs in U to the standard k-dimensional foliation on $\mathbb{R}^n$, the derivative map Dx maps the integrable distribution Δ onto the distribution $(T\mathbb{R}^k) \times \mathbb{R}^{n-k}$:

$$\begin{tikzcd}
\Delta|_U \arrow[r, dashed] \arrow[d, hook] & (T\mathbb{R}^k) \times \mathbb{R}^{n-k} \arrow[d, hook] \\
TU \arrow[r, "Dx"] \arrow[d] & T\mathbb{R}^n \arrow[d] \\
U \arrow[r, "x"] & (-\varepsilon, \varepsilon)^n.
\end{tikzcd}$$

Take the connected component $V \subset \gamma^{-1}(U)$ that contains the time t. This ensures that the image of the map γ evaluated on this component is connected in U.

Consider the smooth composition of maps $x \circ \gamma : V \longrightarrow (-\varepsilon, \varepsilon)^n$. Since the derivative $D\gamma$ maps into the distribution Δ, by chain rule, the derivative of the composition $D(x \circ \gamma)$ maps into $(T\mathbb{R}^k) \times \mathbb{R}^{n-k}$:

$$\begin{tikzcd}
 & \Delta|_U \arrow[r] \arrow[d, hook] & (T\mathbb{R}^k) \times \mathbb{R}^{n-k} \arrow[d, hook] \\
T\mathbb{R}|_V \arrow[ur] \arrow[r, "D\gamma"] \arrow[d] & TU \arrow[r, "Dx"] \arrow[d] & T\mathbb{R}^n \arrow[d] \\
V \arrow[r, "\gamma"] & U \arrow[r, "x"] & (-\varepsilon, \varepsilon)^n.
\end{tikzcd}$$

So, for all $s \in \gamma^{-1}(U)$, there is the linear map

$$D_s(x \circ \gamma) : T_s\mathbb{R} \longrightarrow (T_{x\circ\gamma(s)}\mathbb{R}^k) \times \mathbb{R}^{n-k}.$$

This yields that the derivative $\frac{\partial (x\circ\gamma)^i}{\partial t}(s) = 0$ vanishes for all $i = k+1, \ldots, n$ and $s \in V$. So, the coordinates $(x \circ \gamma)^i$ are constant on the connected set V. As $t \in V$ and

$$(x \circ \gamma)^i(t) = x^i(\gamma(t)) = 0,$$

then $x^i(\gamma(s)) = 0$ for all such s. Thus, for all $s \in V$, the point $\gamma(s)$ is in the same leaf as $\gamma(t)$ in the foliated neighborhood

$$\gamma(s) \in \{p' \in U : x^{k+1}(p') = \ldots x^n(p') = 0\}.$$

That is to say that given any arbitrary point in $\mathbb{R}$, there is a neighborhood of said point that maps entirely into one leaf.

Let $s \in \mathbb{R}$ and consider the interval from 0 to s. Without loss of generality, assume $0 < s$. For each point $t \in [0, s]$, there exists a connected neighborhood V_t on which γ maps said neighborhood into exactly one leaf. Cover $[0, s]$ with such neighborhoods. Since $[0, s]$ is compact, there is a finite subcover. Denote the finitely many neighborhoods $V_0, \dots, V_m$. Label the neighborhood containing 0 V_0 and the neighborhood containing s V_m. Refine the covering such that $V_i \cap V_j \neq \varnothing$ if $|i - j| = 1$.

As γ maps V_0 into a single leaf and $0 \in V_0$, the image of V_0 under γ lies in L. Take $\tau_1 \in V_0 \cap V_1$. Since $\tau_1 \in V_0$, the point $\gamma(\tau_1)$ lies in the leaf L. Since $\gamma|_{V_1}$ maps into a single leaf and $\tau_1 \in V_1$, the image of V_1 is contained in the leaf $\gamma(V_1) \subset L$.

Proceed inductively. Assume γ maps V_i into L. There exists $\tau_{i+1} \in V_i \cap V_{i+1}$ that gets mapped into L. Since γ maps all of V_{i+1} into a single leaf, it must map into L. Each interval is thus mapped into L as must s. s was arbitrary, so γ maps into L. $\square$

Proposition 2.0.34. *Let (M, Δ) be integrable and horizontally path-connected. Then the associated foliation has a single leaf.*

Proof. Let $p, p' \in M$ be arbitrary points. Since (M, Δ) is horizontally path-connected, there exists a path tangent to the integrable distribution $\gamma : [0, 1] \longrightarrow (M, \Delta)$ such that $\gamma(0) = p$ and $\gamma(1) = p'$. By Proposition 2.0.33, the points p and p' must lie in the same leaf. Since these points were arbitrary, all points of M lie in the same leaf of the foliation. $\square$

On the other hand, when the distribution ξ is bracket-generating on the manifold M, by the Chow-Rashevskii Theorem [15], [24], (M, ξ) is horizontally path-connected. Thus, every contact manifold is horizontally path-connected.

This yields further insight into how integrable distributions and bracket-generating distributions are opposites. An integrable distribution yields a foliation of the space on which it is defined, each leaf of which has tangent space agreeing with the distribution. But, it is not possible to move between arbitrary points in the manifold by horizontal paths. Horizontal paths remain in the leaf that they start in.

On the other hand, a bracket-generating distribution makes it impossible to embed a submanifold, even locally, such that the submanifold has tangent space agreeing with the distribution. But, the space is horizontally path-connected.

Heisenberg group as a contact manifold

The standard example of a contact manifold is the Heisenberg group. In fact, as will be noted in the Theorem of Darboux, all contact manifolds are locally modeled by a Heisenberg group.

Example 2.0.35. (**nth Heisenberg Group**) Let $M = \mathbb{R}^{2n+1}$ with coordinates denoted by $x_1, \ldots, x_n, y_1, \ldots, y_n, t$. Define a codimension 1 distribution on $\mathbb{R}^{2n+1}$ by

$$\xi^{std} := \operatorname{span}(X_1, \ldots, X_n, Y_1, \ldots, Y_n),$$

where, for $i = 1, \ldots, n$, the vector fields X_i and Y_i are defined by

$$X_i := \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t} \text{ and } Y_i := \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}.$$

To calculate the Lie Bracket $[X_i, Y_i]$, consider a smooth real-valued function $f : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}$. Then, $X_i Y_i$ acts on f,

$$\begin{aligned} X_i(Y_i f) &= \left(\frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial t}\right)\left(\frac{\partial f}{\partial y_i} - 2x_i\frac{\partial f}{\partial t}\right) \\ &= \frac{\partial^2 f}{\partial x_i \partial y_i} + 2y_i\frac{\partial^2 f}{\partial t \partial y_i} - 2\frac{\partial f}{\partial t} - 2x_i\frac{\partial^2 f}{\partial x_i \partial t} - 4x_i y_i\frac{\partial^2 f}{\partial,^2}t, \end{aligned}$$

and $Y_i X_i$ acts on f:

$$Y_i(X_i f) = \frac{\partial^2 f}{\partial x_i \partial y_i} - 2x_i\frac{\partial^2 f}{\partial x_i \partial t} + 2\frac{\partial f}{\partial t} + 2y_i\frac{\partial^2 f}{\partial t \partial y_i} - 4x_i y_i\frac{\partial^2 f}{\partial t^2}.$$

So, the Lie Bracket $[X_i, Y_i]$ acting on f yields an equality of functions

$$[X_i, Y_i](f) = X_i(Y_i f) - Y_i(X_i f) = -4\frac{\partial f}{\partial t}$$

and, since the function f was arbitrary, there is an equality of vector fields, $[X_i, Y_i] = -4\frac{\partial}{\partial t}$. As $\frac{\partial}{\partial t}$ is a vector field that is not tangent to ξ,

$$T\mathbb{R}^{2n+1} = \xi^{std} \bigoplus \operatorname{span}\left(-4\frac{\partial}{\partial t}\right) = \xi^{std} \bigoplus \operatorname{span}([X_i, Y_i])$$

for any i and ξ^{std} is bracket-generating. This contact manifold is referred to as the *nth Heisenberg group* and is denoted by $\mathbb{H}^n := (\mathbb{R}^{2n+1}, \xi^{std})$.[2]

Now that the contact structure on $\mathbb{H}^n$ has been established and we have a means of embedding this structure into other contact manifolds via distributional embeddings, we can make precise that contact manifolds are locally modeled by the Heisenberg group.

[2]In the case that $n = 1$, the contact manifold can be thought of (erroneously) as possible flight paths of a Golden Eagle. Watch him soar!

Theorem 2.0.36. *(**Theorem of Darboux** [5]) Let (M, ξ) be a $(2n+1)$-dimensional contact manifold. For any point $p \in M$, there exists an open distributional embedding*

$$\varphi : \mathbb{H}^n \hookrightarrow (M, \xi)$$

of the nth Heisenberg group into the contact manifold (M, ξ) such that $\varphi(0) = p$.

The neighborhood $\varphi(\mathbb{H}^n)$ of the point p will be referred to as a *Darboux neighborhood* of p. When it presents no extra confusion, Darboux neighborhood will also refer to the open distributional embedding.

Horizontal paths in $\mathbb{H}^1$

For the first Heisenberg group $\mathbb{H}^1$, horizontal paths are completely recovered from their projections onto the xy-plane, as the next example demonstrates.

Example 2.0.37. (Determining horizontal paths in $\mathbb{H}^1$) Let

$$\gamma = (\gamma_1, \gamma_2, \gamma_3) : [a, b] \longrightarrow \mathbb{H}^1$$

be a horizontal path in the first Heisenberg group. Here, γ_i is the projection of the path γ onto the ith coordinate of $\mathbb{R}^3$.

Recall that the associated distribution ξ^{std} is spanned globally by the vector fields

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t} \text{ and } Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}$$

where $\mathbb{R}^3$ has global coordinates given by x, y, and t. As the path γ is horizontal, its derivative $D\gamma$ must map into the contact distribution ξ^{std};

$$\begin{tikzcd}
 & \xi^{std} \arrow[d, hook] \\
T[a,b] \arrow[r, "D\gamma"] \arrow[ur, dashed, "\exists"] \arrow[d] & T\mathbb{R}^3 \arrow[d] \\
[a,b] \arrow[r, "\gamma"] & \mathbb{R}^3.
\end{tikzcd}$$

Fixing the standard Euclidean metric on $[a,b]$, the unit vector $1_s \in T_s[a,b]$ is identified in the tangent space of each point $s \in [a,b]$. As $D\gamma$ is point-wise a linear map from the 1-dimensional tangent space spanned by the unit vector, $D_s\gamma$ is determined by how it acts on 1_s for each $s \in [a,b]$.

Denote the coordinate-wise derivatives of γ with respect to s by γ_i' for $i = 1,2,3$ where, for each $s \in [a,b]$,

$$(\gamma_1'(s), \gamma_2'(s), \gamma_3'(s)) = D_s\gamma(1_s).$$

As $(\gamma_1'(s), \gamma_2'(s), \gamma_3'(s))$ yields a vector in $\xi^{std}_{\gamma(s)} = \text{span}\{X_{\gamma(s)}, Y_{\gamma(s)}\}$ for each $s \in [a,b]$, and as $(\gamma_1', \gamma_2', \gamma_3')$ smoothly-varies over $[a,b]$, there exists smooth coefficient functions $a_\gamma, b_\gamma : [a,b] \to \mathbb{R}$ such that

$$(\gamma_1'(s), \gamma_2'(s), \gamma_3'(s)) = a_\gamma(s) X_{\gamma(s)} + b_\gamma(s) Y_{\gamma(s)}$$

for each $s \in [a,b]$.

Thus, the following vectors are equal in $\xi^{std}_{\gamma(s)} = \text{span}\{X_{\gamma(s)}, Y_{\gamma(s)}\}$ for each $s \in$

$[a,b]$:

$$\begin{aligned}(\gamma_1'(s),\gamma_2'(s),\gamma_3'(s)) &= a_\gamma(s)X_{\gamma(s)} + b_\gamma(s)Y_{\gamma(s)}\\ &= (a_\gamma(s), b_\gamma(s), 2(\gamma_2(s)a_\gamma(s) - \gamma_1(s)b_\gamma(s))),\end{aligned}$$

In fact, these coordinate functions are determined completely by this equality as

$$a_\gamma(s) = \gamma_1'(s) \text{ and } b_\gamma(s) = \gamma_2'(s)$$

and therefore, γ_3' is determined by information about γ_1 and γ_2:

$$\gamma_3'(s) = 2(\gamma_2(s)\gamma_1'(s) - \gamma_1(s)\gamma_2'(s)).$$

Thus, given a smooth path $(\gamma_1,\gamma_2) : [a,b] \longrightarrow \mathbb{R}^2$ and a specified initial t coordinate $\gamma_3(a)$, there is a unique lift to a horizontal path

$$\gamma = (\gamma_1,\gamma_2,\gamma_3) : [a,b] \longrightarrow \mathbb{H}^1$$

where γ_3 is determined by

$$\gamma_3(s) = \gamma_3(a) + 2\int_a^s \gamma_2(u)\gamma_1'(u) - \gamma_1(u)\gamma_2'(u)\ du. \tag{$\star$}$$

If we additionally ask that γ is a closed curve, that is $\gamma(a) = \gamma(b)$, then $(\star)$ yields that

$$0 = \gamma_3(b) - \gamma_3(a) = \int_a^b \gamma_2(s)\gamma_1'(s) - \gamma_1(s)\gamma_2'(s)\ ds.$$

Let A denote the region bounded by the closed curve (γ_1,γ_2) in $\mathbb{R}^2$ running from time

a to time b. Then the signed area of A is given by

$$\int_A dx \wedge dy = \frac{1}{2}\int_{\partial A} xdy - ydx = -\frac{1}{2}\int_a^b \gamma_2(s)\gamma_1'(s) - \gamma_1(s)\gamma_2'(s)\ ds = 0.$$

Here, the first equality is guaranteed by Stokes' theorem and the second by change of variables. Thus, the region enclosed by the closed path (γ_1, γ_2) must have signed area zero. Furthermore, horizontal loops in $\mathbb{H}^1$ are generated by closed loops in $\mathbb{R}^2$ that bound signed area zero.

Examples of horizontal paths in $\mathbb{H}^1$

Example 2.0.38. The smooth map $\iota : [a,b] \longrightarrow \mathbb{H}^1$ given by $\iota(t) := (t,0,0)$ is a horizontal path. Indeed, given any $t_0 \in [a,b]$,

$$D_{t_0}\iota(1_{t_0}) = \frac{\partial \iota}{\partial t}(t_0) = \left(\frac{\partial}{\partial t}\bigg|_{t=t_0}(t), 0, 0\right) = (1,0,0)_{(t_0,0,0)} = \frac{\partial}{\partial x}\bigg|_{(t_0,0,0)} \in \xi^{std}_{\iota(t_0)}.$$

We now construct an example of a horizontal path γ that has infinitely many critical points, i.e., points where the first derivative of γ vanishes.

This is relevant to this dissertation as the map γ can then be used to construct horizontal maps out of spheres of dimension greater than 1. For these newly constructed maps, the set on which the the derivative is the zero will have infinitely many components. In the chapter discussing the higher horizontal homotopy groups of contact 3-manifolds, this will show that the trees constructed in Lemma 5.0.40 can potentially have infinitely many vertices.

A few results and tools are established first before constructing the desired map. Let $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ be a non-decreasing smooth map such that $\rho(0) = 0$, $\rho(1) = 1$ and $(\rho')^{-1}\{0\} = \mathbb{R} \setminus (0,1)$.

Let $s > 0$ be some positive number, thought of as a time. Define a smooth map

$\rho_s : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\rho_s(t) := s\rho\left(\frac{1}{s}t\right).$$

Thus, ρ_s reparametrizes ρ so that it bijectively maps $[0,s]$ onto itself, rather than $[0,1]$ onto itself.

Observation 2.0.39. For non negative integer k,

$$\frac{\partial^k}{\partial t^k}(\rho_s(t)) = s^{1-k}\rho^{(k)}\left(\frac{1}{s}t\right)$$

where $\rho^{(k)}$ is the kth derivative of ρ.

Observation 2.0.40. The maximum of $\left|\rho^{(k)}\left(\frac{1}{s}t\right)\right|$ does not depend on s. Indeed, since ρ is constant outside of $[0,1]$, for $k > 0$, the kth derivative $\rho^{(k)}$ is 0 outside of $[0,1]$ as is $\rho^{(k)}\left(\frac{1}{s}(-)\right)$ outside of $[0,s]$. Since multiplication by $\frac{1}{s}$ is a diffeomorphism from $[0,s]$ to $[0,1]$,

$$\max_{t\in[0,s]}\left|\rho^{(k)}\left(\frac{1}{s}t\right)\right| = \max_{t\in[0,1]}\left|\rho^{(k)}(t)\right| =: M_k.$$

Since $\rho^{(k)}$ is continuous, M_k is finite.

We now construct a family of horizontal paths into $\mathbb{H}^1$ inspired by the flow calculation of $[X,Y]$. See [31] for a description of the flow definition of Lie brackets. By using ρ_s, these "flows" can be glued together to form a horizontal helix.

Example 2.0.41. Let $s > 0$ be some fixed amount of time, $h \in \mathbb{R}$ be an initial height, and $C \neq 0$ be some constant. Define horizontal paths $\gamma_i^{s,h,C} : [0,s] \longrightarrow \mathbb{H}^1$, $i = 1,2,3,4$ as follows:

$$\begin{aligned}
\gamma_1^{s,h,C}(t) &:= (Ct, 0, h)\\
\gamma_2^{s,h,C}(t) &:= (Cs, Ct, h - C^2st)\\
\gamma_3^{s,h,C}(t) &:= (Cs - Ct, Cs, h - C^2s^2)\\
\gamma_4^{s,h,C}(t) &:= (0, Cs - Ct, h - C^2s^2)
\end{aligned}$$

These maps are indeed horizontal, as can be verified from their derivatives:

$$\begin{aligned}
&\tfrac{\partial}{\partial t}(\gamma_1^{s,h,C}(t)) = (C, 0, 0) = CX \in \xi^1\\
&\tfrac{\partial}{\partial t}(\gamma_2^{s,h,C}(t)) = (0, C, -C^2s) = C(\tfrac{\partial}{\partial y} - Cs\tfrac{\partial}{\partial z}) = C(\tfrac{\partial}{\partial y} - x(\gamma_2^{s,h,C}(t))\tfrac{\partial}{\partial z}) = CY \in \xi^1\\
&\tfrac{\partial}{\partial t}(\gamma_3^{s,h,C}(t)) = (-C, 0, 0) = -CX \in \xi^1\\
&\tfrac{\partial}{\partial t}(\gamma_4^{s,h,C}(t)) = (0, -C, 0) = -C(\tfrac{\partial}{\partial y} - 0\tfrac{\partial}{\partial z}) = -C(\tfrac{\partial}{\partial y} - x(\gamma_4^{s,h,C}(t))\tfrac{\partial}{\partial z}) = -CY \in \xi^1
\end{aligned}$$

Notice that for $i = 1, 2, 3$, $\gamma_i^{s,h,C}(s) = \gamma_{i+1}^{s,h,C}(0)$. So, these can be glued together back-to-front in order to gain a continuous map from $[0, 4s]$ into $\mathbb{R}^3$. In order to make this map smooth, precompose each $\gamma_i^{s,h,C}$ by ρ_s before gluing them together. Since all derivative of ρ_s vanish outside of $(0, s)$, this will yield a smooth map:

$$\gamma^{s,h,C} : [0, 4s] \longrightarrow \mathbb{H}^1$$

$$\gamma^{s,h,C}(t) := \begin{cases}
\gamma_1^{s,h,C} \circ \rho_s(t) & 0 \le t \le s\\
\gamma_2^{s,h,C} \circ \rho_s(t - s) & s \le t \le 2s\\
\gamma_3^{s,h,C} \circ \rho_s(t - 2s) & 2s \le t \le 3s\\
\gamma_4^{s,h,C} \circ \rho_s(t - 3s) & 3s \le t \le 4s.
\end{cases}$$

So $\gamma^{s,h,C}$ is a horizontal path with initial position $(0, 0, h)$, runs for time $4s$, and ends at $(0, 0, h - C^2s^2)$. Also, all derivatives of $\gamma^{s,h,C}$ vanish at $t = 0, s, 2s, 3s, 4s$. Thus, such paths can be glued together to form horizontal paths.

Example 2.0.42. Fix a sequence (s_n) in $(0,1]$ such that the series $\sum_{n=1}^{\infty} s_n$ converges. Define a continuous path γ that is horizontal everywhere but possibly its terminal time.

Choose an initial height h_1 so that the initial position is $(0,0,h_1)$. Let $\gamma^1 : [0,s_1] \longrightarrow \mathbb{H}^1$ be the horizontal path $\gamma^1 := \gamma^{s_1,h_1,s_1}$. The end position of γ^1 is

$$\gamma^1(s_1) = (0,0,h_1 - (s_1)^2(s_1)^2) = (0,0,h_1 - s_1^{2(1)+2}).$$

Define $\gamma^2 : [0,s_2] \longrightarrow \mathbb{H}^1$ so that $\gamma^2(0) = \gamma^1(s_1)$. Thus, take $h_2 = h_1 - s_1^{2(1)+2}$ and define $\gamma^2 := \gamma^{s_2,h_2,s_2^2}$.

Proceed inductively. For each s_n, define a horizontal path $\gamma^n : [0,s_n] \longrightarrow \mathbb{H}^1$ by $\gamma^n := \gamma^{s_n,h_n,s_n^n}$ where $h_n = h_1 - \sum_{i=1}^{n-1} s_i^{2i+2}$. This is chosen for h_n so that, for each n, $\gamma^n(s_n) = \gamma^{n+1}(0)$ and γ^n and γ^{n+1} can be glued together to result in a continuous, in fact smooth, path.

Let $\gamma : [0, 4\sum_{i=1}^{\infty} s_i) \longrightarrow \mathbb{H}^1$ be the smooth map that results from this gluing. For $t \in [4\sum_{i=1}^{n-1} s_i, 4\sum_{i=1}^{n} s_i]$, define $\gamma(t) := \gamma^n(t - 4\sum_{i=1}^{n-1} s_i)$.

We can ask if γ continuously extends to $t = 4\sum_{i=1}^{\infty} s_i < \infty$. Since $\sum_{i=1}^{\infty} s_i$ converges and eventually all s_i are less than 1, $\sum_{i=1}^{\infty} s_i^{2i+2}$ converges by comparison test. Then, the sequence of heights converges, $\lim_{n\to\infty} h_n = h_1 - \sum_{i=1}^{\infty} s_i^{2i+2} < \infty$, and we can continuously extend γ:

$$\gamma\left(4\sum_{i=1}^{\infty} s_i\right) := h_1 - \sum_{i=1}^{\infty} s_i^{2i+2}.$$

So $\gamma : [0, 4\sum_{i=1}^{\infty} s_i] \longrightarrow \mathbb{R}^3$ is continuous and is horizontal into $\mathbb{H}^1$ everywhere but possibly $t = 4\sum_{i=1}^{\infty} s_i$. Following a few propositions, it will be shown that γ is in fact horizontal on the entirety of its domain.

Proposition 2.0.43. *Let $\gamma : (c,d) \longrightarrow \mathbb{R}^n$ be a continuous map such that for some $t_0 \in (c,d)$, $\gamma|_{(c,d)\smallsetminus\{t_0\}}$ is smooth. If for each $k > 0$, $\lim_{t\to t_0} \gamma^{(k)}(t) = L_k$ exists, then γ is*

smooth and $\gamma^{(k)}(t_0) = L_k$.

Proof. It is enough to argue that this is true when $(c,d) = \mathbb{R}$ and $t_0 = 0$.

Suppose $n = 1$. First, take $k = 1$ for a base case. By definition,

$$\gamma'(0) = \lim_{h\to 0} \frac{\gamma(h) - \gamma(0)}{h}.$$

For the moment, consider the limit as h approaches 0 from above. For any $h > 0$, since γ is continuous on $[0,h]$ and differentiable on $(0,h)$, by Mean Value Theorem, there exists $\theta(h) \in (0,h)$ such that $\gamma'(\theta(h)) = \frac{\gamma(h)-\gamma(0)}{h}$. By squeeze theorem, $\theta(h) \to 0$ as $h \to 0^+$. Thus, since $\lim_{t\to 0}\gamma'(t)$ exists,

$$\lim_{h\to 0^+} \frac{\gamma(h) - \gamma(0)}{h} = \lim_{h\to 0^+} \gamma'(\theta(h)) = L_1.$$

The same result holds if h approaches 0 from below. Thus, $\gamma'(0) = \lim_{h\to 0}\frac{\gamma(h)-\gamma(0)}{h} = L_1$ and γ' is continuous.

Proceed by induction. Suppose $\gamma^{(k-1)}$ is continuous. By assumption, $\gamma^{(k-1)}$ is differentiable away from 0. Thus, the base case can be repeated with $\gamma^{(k-1)}$ replacing every instance of γ to yield that $\gamma^{(k)}(0) = \lim_{t\to 0}\gamma^{(k)}(t) = L_k$ and $\gamma^{(k)}$ is continuous. By induction, $\gamma^{(k)}$ exists and is continuous for all k. So, γ is smooth.

For a general n, $\gamma = (\gamma_1, \ldots, \gamma_n)$ where $\gamma_i : \mathbb{R} \longrightarrow \mathbb{R}$. Since γ satisfies the conditions of the hypothesis, so does each γ_i where $n = 1$. The result follows from $\gamma^{(k)} = (\gamma_1^{(k)}, \ldots, \gamma_n^{(k)})$ and the $n = 1$ case above.

□

Corollary 2.0.44. *Let* $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ *be a continuous map that is smooth on* (a,b). *If for each* $k > 0$

$$\lim_{t\to a^+} \gamma^{(k)}(t) = 0 = \lim_{t\to b^-} \gamma^{(k)}(t),$$

then γ is smooth on its domain and all of its derivatives vanish at a and b.

Proof. In order to show that γ is smooth at a and b, it is enough to show that γ extends to a smooth map $\bar{\gamma} : \mathbb{R} \longrightarrow \mathbb{R}^n$. For $t \leq a$, set $\bar{\gamma}(t) = \gamma(a)$ and for $t \geq b$, set $\bar{\gamma}(t) = \gamma(b)$.

It is immediate that $\bar{\gamma}$ is continuous. Also, since $\bar{\gamma}$ is constant on $(-\infty, a)$, $\bar{\gamma}$ is smooth everywhere but possibly $t = a$. So, $\bar{\gamma}|_{(-\infty,b)}$ is continuous, smooth everywhere but possibly a, and $\lim_{t\to a} \bar{\gamma}^{(k)}(t) = 0$ for all $k > 0$. By Proposition 2.0.43, $\bar{\gamma}$ is smooth at a. A similar argument yields that it is also smooth at b. So, a smooth extension of γ exists.

□

Proposition 2.0.45. *The map γ described in Example 2.0.42 is horizontal:*

$$\gamma : \left[0, 4\sum_{i=1}^{\infty} s_i\right] \longrightarrow \mathbb{H}^1.$$

Proof. By Corollary 2.0.44, it is enough to show that for every $k > 0$,

$$\lim_{t\to 4\sum_{i=1}^{\infty} s_i} \gamma^{(k)}(t) = 0.$$

Since k is arbitrary, γ is smooth. The map is horizontal into $\mathbb{H}^1$ since its first derivative vanishes at $t = 4\sum_{i=1}^{\infty} s_i$ (Observation 2.0.11) and it has already been established that γ is horizontal everywhere else.

Fix $k > 0$. For ease of notation, let $S_{n,j} := js_n + 4\sum_{i=1}^{n} s_i$ for $j = 0, 1, 2, 3$. Using the description of γ^n in Example 2.0.41 and Observation 2.0.39, write down a description of the kth derivative of γ for $t \in [S_{n-1,0}, S_{n,0}]$.

$$\gamma^{(k)}(t) := \begin{cases} \left(s_n^{n+1-k}\rho^{(k)}\left(\frac{1}{s_n}\left(t - S_{n-1,0}\right)\right), 0, 0\right), \\ \hfill \text{if } S_{n-1,0} \le t \le S_{n-1,1} \\ \left(0, s_n^{n+1-k}\rho^{(k)}\left(\frac{1}{s_n}\left(t - S_{n-1,1}\right)\right), -s_n^{2n+2-k}\rho^{(k)}\left(\frac{1}{s_n}\left(t - S_{n-1,1}\right)\right)\right), \\ \hfill \text{if } S_{n-1,1} \le t \le S_{n-1,2} \\ \left(-s_n^{n+1-k}\rho^{(k)}\left(\frac{1}{s_n}\left(t - S_{n-1,2}\right)\right), 0, 0\right), \\ \hfill \text{if } S_{n-1,2} \le t \le S_{n-1,3} \\ \left(0, -s_n^{n+1-k}\rho^{(k)}\left(\frac{1}{s_n}\left(t - S_{n-1,3}\right)\right), 0\right), \\ \hfill \text{if } S_{n-1,3} \le t \le S_{n,0}. \end{cases}$$

For $t \in [S_{n-1,1}, S_{n-1,2}]$,

$$\left\|\gamma^{(k)}(t)\right\|^2 = \left(s_n^{2n+2-2k} + s_n^{4n+4-2k}\right)\left[\rho^{(k)}\left(\frac{1}{s_n}\left(t - S_{n-1,1}\right)\right)\right]^2.$$

By Observation 2.0.40, since $t - S_{n-1,1}$ ranges between 0 and s_n,

$$\left[\rho^{(k)}\left(\frac{1}{s_n}\left(t - S_{n-1,1}\right)\right)\right]^2 \le M_k^2$$

and $\|\gamma^{(k)}(t)\|^2 \le \left(s_n^{2n+2-2k} + s_n^{4n+4-2k}\right) M_k^2$.

Let $\varepsilon > 0$ be given. Since $s_n \to 0$ as $n \to \infty$ and $s_n \in (0, 1]$, $(s_n^{2n+2-2k} + s_n^{4n+4-2k})M_k^2$ limits to 0 as $n \to \infty$. So, there exists $N \in \mathbb{N}$ such that, for all $n \ge N$,

$$\left\|\gamma^{(k)}(t)\right\|^2 \le \left(s_n^{2n+2-2k} + s_n^{4n+4-2k}\right) M_k^2 < \varepsilon^2.$$

Thus, $\|\gamma^{(k)}(t)\| < \varepsilon$ for $S_{N-1,0} < S_{n-1,1} \le t \le S_{n-1,2}$ for some $n > N$.

Now, let $S_{N-1,0} < S_{n-1,j} \leq t \leq S_{n-1,j}$ for $j = 0,2,3$, $n > N$. Then, using Observation 2.0.40,

$$\left\|\gamma^{(k)}(t)\right\|^2 = s_n^{2n+2-2k}\left[\rho^{(k)}\left(\frac{1}{s_n}\left(t - S_{n-1,j}\right)\right)\right]^2 < \left(s_n^{2n+2-2k} + s_n^{4n+4-2k}\right) M_k^2 < \varepsilon^2$$

Therefore, for $t \in (S_{N-1,0}, 4\sum_{i=1}^{\infty} s_i)$, $\|\gamma^{(k)}(t)\| < \varepsilon$.

□

Thus, γ is a horizontal map whose first derivative vanish on a set of infinite cardinality, $\{S_{n,j}\}_{n\in\mathbb{N}}^{j=0,1,2,3} \cup \{4\sum_{i=1}^{\infty} s_i\} \subset [0, 4\sum_{i=1}^{\infty} s_i]$.

We end this section with an example of a horizontal map into the first Heisenberg group $\mathbb{H}^1$ from the 2-sphere whose set of points where the derivative vanishes has infinitely many components.

Example 2.0.46. Define a horizontal map $\gamma : [0,2] \to \mathbb{H}^1$ as in Example 2.0.42 with respect to the sequence $(s_n) = \left(\frac{1}{2^n}\right)$ and the height $h_1 = 0$. The subset of critical points of the map γ is the infinite set

$$\{0,2\} \bigcup \left\{\frac{k}{4} \cdot \frac{1}{2^n}\right\}_{n\in\mathbb{N}}^{k=1,2,3,4}.$$

Define a horizontal map out of the 2-sphere via the composition

$$\gamma \circ (\mathsf{pr} + 1) : \mathbb{S}^2 \longrightarrow \mathbb{H}^1$$

where $\mathsf{pr} : \mathbb{S}^2 \to [-1,1]$ is the projection of the 2-sphere onto its last coordinate. This map is indeed horizontal as γ is horizontal (Lemma 2.0.8). The subspace of points where the derivative of this composition vanishes is a collection of latitudes on the

2-sphere as well as the north and south pole:

$$\left\{(x,y)\in\mathbb{S}^2\ :\ y\in\{-1,1\}\bigcup\left\{\frac{k}{4}\cdot\frac{1}{2^n}-1\right\}_{n\in\mathbb{N}}^{k=1,2,3,4}\right\}.$$

Note that this subspace has infinitely many connected components.

.

Key Observation about horizontal maps into contact 3-manifolds

This dissertation is primarily focused on studying the contact structure of contact 3-manifolds. By the Theorem of Darboux, these spaces all are locally the first Heisenberg group $\mathbb{H}^1$ Our primary means of studying these structures will be horizontal maps. The following key observation indicates that horizontal maps into a contact 3-manifold are rather restrictive. More specifically, the key observation states that horizontal maps into a contact 3-manifold (M,ξ) have rank at most 1.

Lemma 2.0.47. *[Key Observation] Let $f: N \longrightarrow (M,\xi)$ be a horizontal map from a manifold N into a contact 3-manifold (M,ξ). Then the map f has rank at most 1.*

Proof. Let $f: N\to(M,\xi)$ be a horizontal map with domain N, an n-dimensional manifold. Choose a point $p\in N$. Let $U\subset N$ be a neighborhood with a local parameter $x: U\to\mathbb{R}^n$. By the Theorem of Darboux (Theorem 2.0.36), there exists an open distributional embedding $\varphi:\mathbb{H}^1\hookrightarrow(M,\xi)$ such that $\varphi(0)=f(p)$. Thus, locally, the map f is a horizontal map from a $\mathbb{R}^n$ to $\mathbb{H}^1$:

$$\begin{tikzcd}
N \arrow[r, "f"] & (M,\xi) \\
U \arrow[u, hook] \arrow[r, "f|"] \arrow[d, "x"] & \varphi(\mathbb{H}^1) \arrow[u, hook] \\
\mathbb{R}^n \arrow[r, dashed, "\varphi^{-1}\circ f|\circ x^{-1}"] & \mathbb{H}^1. \arrow[u, "\varphi"]
\end{tikzcd}$$

Lemma 4.13 in [6] states that horizontal maps from Euclidean space into $\mathbb{H}^1$ have rank at most 1. Thus, the horizontal map $\varphi^{-1} \circ f| \circ x^{-1}$ has rank at most 1. Therefore, the horizontal map f has rank at most 1 on U. Since the point p was arbitrary, f has rank at most 1 on its entire domain.

□

Remark 2.0.48. The proof of Lemma 4.13 in [6] uses that Legendrian submanifolds (horizontal embeddings) of $\mathbb{H}^1$ have dimension at most 1, which is true for all contact 3-manifolds. Thus, Lemma 2.0.47 can be proved without appealing to the Theorem of Darboux by inspecting dimensional constraints of Legendrian submanifolds within contact 3-manifolds.

Sub-Riemannian manifolds

It is of interest to try to better understand contact manifolds through techniques of metric topology. As is indicated in Lemma 3.0.51, a contact manifold determines a sheaf on the smooth site. As will be argued in this dissertation, this sheaf is not representable (Corollary 4.0.24). But, with considerations of metric topology, there is a sheaf determined by a contact manifold that is representable by a metric space.

Before proceeding, we must endow these spaces with a metric since contact manifolds do not have an inherent metric structure. Thus, an appropriate notion of a smoothly-varying inner product on a contact manifold (a sub-Riemannian metric) will be defined, followed shortly thereafter by an associated idea of a metric (a Carnot-Carathéodory metric). This new metric topology on a contact manifold will then be probed via Lipschitz maps.

All smooth manifolds can be endowed with a Riemannian metric which gives rise to a path metric on the space. This path metric is defined to be the infimum

of the lengths of all smooth paths between points. Though we can endow a contact manifold with such a metric, this structure will be ignorant to the contact distribution and thus provide no useful information about the contact structure.

We slightly adjust the path-metric in order to be sensitive to the contact structure. Rather than beginning with a Riemannian metric, which provides an inner product on each tangent space, we can ask instead for a *sub-Riemannian metric* that provides a smoothly-varying inner product only on the contact distribution. After replacing a Riemannian metric by a sub-Riemannian metric, the associated path metric, which is called the Carnot-Carathéodory metric, will feel the contact structure.

Definition 2.0.49. A *sub-Riemannian manifold* is a triple (M, ξ, g) consisting of

- a connected manifold M,
- a distribution $\xi \subset TM$, and
- a smooth map

$$g : \xi \times_M \xi \longrightarrow \mathbb{R},$$

  for which, for each $p \in M$, the map $g_p : \xi_p \oplus \xi_p \to \mathbb{R}$ is an inner product on the vector space ξ_p

such that ξ is bracket-generating. Such a g is referred to as a *sub-Riemannian metric* on (M, ξ).

Sub-Riemannian manifolds are more general than contact manifolds. We do not stipulate that the dimension of the underlying manifold is odd nor that the distribution is codimension 1. For example, any manifold M with a Riemannian metric is an example of a sub-Riemannian manifold, where $\xi = TM$ is trivially bracket-generating.

Any smooth manifold with a bracket-generating distribution can be endowed with a sub-Riemannian structure. As noted above, any manifold can be endowed with a Riemannian metric. Simply restricting the Riemannian metric to the bracket-generating distribution yields a sub-Riemannian metric. Going forward, we can assume that any manifold with a bracket-generating distribution has a sub-Riemannian structure. We now make specific reference to the standard sub-Riemannian structure on the nth Heisenberg group $\mathbb{H}^n$.

Example 2.0.50. Continuing from Example 2.0.35, the contact manifold $\mathbb{H}^n$ is naturally endowed with a sub-Riemannian structure. Indeed, define a sub-Riemannian metric g such that, for each $p \in \mathbb{H}^n$, the vectors

$$X_1(p), \ldots, X_n(p), Y_1(p), \ldots, Y_n(p)$$

are an orthonormal basis for ξ_p^{std}.

Convention 2.0.51. It will be assumed that $\mathbb{H}^n$ has this sub-Riemannian structure.

Carnot-Carathéodory metric

In order to make sense of a path metric for a sub-Riemannian manifold, a notion of length of curve will be defined. Since a sub-Riemannian metric is only defined on the associated distribution, we only consider paths who are tangent to the distribution, i.e., horizontal paths. Much in the same way that a Riemannian metric is used to define a notion of length of a curve, a sub-Riemannian metric defines a notion of length of horizontal curve.

Definition 2.0.52. Let (M, ξ, g) be a sub-Riemannian manifold. The *Carnot-*

Carathéodory length of a horizontal path $\gamma : [a,b] \longrightarrow (M,\xi)$ is given by

$$l^M(\gamma) := \int_a^b \sqrt{g(\dot{\gamma}(t),\dot{\gamma}(t))}\, dt.$$

The value $g(\dot{\gamma}(t),\dot{\gamma}(t))$ exists for all $t \in [a,b]$ since the path γ is horizontal: $\dot{\gamma}(t) \in \xi_{\gamma(t)}$. Thus, the Carnot-Carathéodory length of γ is well-defined.

As the definition of Carnot-Carathéodory length of a path differs only cosmetically from the Riemannian notion of length of a path, some basic properties in the Riemannian case translate over to the sub-Riemannian case.

Lemma 2.0.53. *Let $\gamma : [a,b] \longrightarrow (M,\xi)$ be a horizontal curve in the sub-Riemannian manifold (M,ξ,g). For any a', b' such that $a \leq a' < b' \leq b$,*

$$l^M(\gamma|_{[a',b']}) \leq l^M(\gamma).$$

Proof.

$$\begin{aligned} l^M(\gamma|_{[a',b']}) &= \int_{a'}^{b'} \sqrt{g\left(\dot{\gamma}(t),\dot{\gamma}(t)\right)}\, dt \\ &\leq \int_a^b \sqrt{g\left(\dot{\gamma}(t),\dot{\gamma}(t)\right)}\, dt \\ &= l^M(\gamma). \end{aligned}$$

□

With a notion of length of a path in a sub-Riemannian manifold, we are now prepared to define a notion of path metric for sub-Riemannian manifolds.

Definition 2.0.54. Let (M,ξ,g) be a sub-Riemannian manifold. The *Carnot-*

Carathéodory metric on M is given by

$$d_{CC}^M(p,p') := \inf\left\{ l^M(\gamma) : \gamma : [a,b] \xrightarrow{\text{horizontal}} (M,\xi), \gamma(a) = p, \gamma(b) = p' \right\}.$$

It is important to note that the supremum is defined over horizontal paths between points rather than arbitrary smooth paths. As such, even in the case where the sub-Riemannian metric is defined as restriction of a Riemannian metric, we do not expect this metric to agree with the Riemannian path metric.

It is well-known that d_{CC}^M is a metric on a connected manifold M provided the space has a sub-Riemannian structure. Indeed, via Chow-Rashevskii Theorem, as the associated distribution is bracket-generating, any two points in the space can be joined by a horizontal path. As such, there is an upper bound on the lengths of paths between any two points and the metric $d_{CC}^M(p,p')$ is defined for all $p,p' \in M$.

For a sub-Riemannian manifold (M,ξ,g), a point $p \in M$, and a non-negative real number r, denote the open ball centered at p of radius r with respect to the Carnot-Carathéodory metric by $B_{CC}^M(p,r)$.

Definition 2.0.55. A set $A \subset M$ is *bounded* if there exists a point $p \in M$ and a non-negative real number $r > 0$ such that

$$A \subset B_{CC}^M(p,r).$$

As previously mentioned, we can assume any manifold with a bracket-generating distribution has a sub-Riemannian structure. Further, we can assume the space is endowed with the associated Carnot-Carathéodory metric d_{CC}^M. $\mathbb{H}^n$ also has a Carnot-Carathéodory metric $d_{CC}^{\mathbb{H}^n}$ with respect to the sub-Riemannian structure detailed in Example 2.0.50.

Lipschitz and locally Lipschitz maps

Now that we have established a preferred notion of a metric on a contact manifold, we will define a means of mapping that is sensitive to this structure.

Definition 2.0.56. Let (X, d^X) and (Y, d^Y) be metric spaces. A map $\varphi : X \longrightarrow Y$ is *Lipschitz* if there exists $L \geq 1$ such that for all $x, x' \in X$

$$d^Y(\varphi(x), \varphi(x')) \leq L\ d^X(x, x').$$

Denote the collection of all Lipschitz maps from (X, d^X) to (Y, d^Y) by

$$\mathsf{Map}^{\mathsf{Lip}}((X, d^X), (Y, d^Y)).$$

Often, we will consider a slightly weaker class of maps; locally Lipschitz maps.

Definition 2.0.57. A map $\varphi : X \longrightarrow Y$ is *locally Lipschitz* if, for all $p \in X$, there exists an open neighborhood $p \in U \subset X$ such that $\varphi|_U$ is Lipschitz.

This allows for a metric discussion about smooth maps. Smooth maps between Riemannian manifolds are locally Lipschitz, but they need not be Lipschitz. For example, consider the smooth self-map of $\mathbb{R}$, $\varphi(x) = x^3$.

As will be shown in a moment, all horizontal maps defined on a Riemannian manifold are locally Lipschitz. This will be a direct corollary to a result that appears in Chapter 6. For convenience, the result is stated now.

Lemma 2.0.58 (Lemma 6.0.2)**.** *Let $f : (M, \xi, g) \longrightarrow (M', \xi', g')$ be a distributional map between sub-Riemannian manifolds. Let $A \subset M$ be a compact set. Then, there exists $B > 0$ such that, for any horizontal path $\gamma : [a, b] \longrightarrow (M, \xi)$ mapping into A,*

the length of the horizontal path $f \circ \gamma$ is bounded:

$$l^{M'}(f \circ \gamma) \leq B \; l^M(\gamma).$$

The following argument uses the existence of strongly convex neighborhoods. A neighborhood is *strongly convex* if for each pair of points in the closure of the neighborhood, there is a unique minimizing geodesic joining the two points, where the interior of the path lies within the neighborhood. For a review of this and other topics in Riemannian geometry, see [7].

Lemma 2.0.59. *Let (M, g) be a Riemannian manifold and let (M', ξ', g') be a sub-Riemannian manifold. Let*

$$f : (M, g) \longrightarrow (M', \xi', g')$$

be a horizontal map. The horizontal map f is locally Lipschitz with respect to the path metric on the domain and the Carnot-Carathéodory metric on the target.

Proof. Fix a point $p \in M$. There is a strongly convex open neighborhood $A \subset M$ of the point p (Proposition 4.2 in [7]). Also, the closure of the set A is compact. We will argue that the map f is Lipschitz on the open subset A.

Take two points $q, q' \in A$. Since the open subset A is strongly convex, there exists a unique minimizing geodesic $\gamma : I \to A$ joining the points q and q'. Thus, the distance between the points with respect to the length metric on M is the length of the geodesic γ:

$$l^M(\gamma) = d^M(q, q').$$

Consider the horizontal path $f \circ \gamma$ in M' joining the points $f(q)$ and $f(q')$. Since the closure of A is compact, by Lemma 6.0.2, the Carnot-Carathéodory length of the

path $f \circ \gamma$ in M' is bounded (up to a scalar B) by the length of the path γ in M:

$$l^{M'}(f \circ \gamma) \leq B\ l^{M}(\gamma).$$

Finally, the Carnot-Carathéodory distance between the points $f(q)$ and $f(q')$ is defined as an infimum over all horizontal paths joining these two points. Thus, we obtain the following inequality:

$$d_{CC}^{M'}(f(q), f(q')) \leq l^{M'}(f \circ \gamma) \leq B\ l^{M}(\gamma) = B\ d^{M}(q, q').$$

Therefore, f is Lipschitz on the open subset A. Since the point p was arbitrary, the map f is Lipschitz. $\square$

Lemma 2.0.60. *Let (M, g) and (M', g') be a Riemannian manifolds and let*

$$f : (M, g) \longrightarrow (M', g')$$

be a smooth map. The smooth map f is locally Lipschitz with respect to the path metric on the domain and the target.

Proof. This is exactly Lemma 2.0.59 where the sub-Riemannian target is taken to be a Riemannian manifold. That is, the bracket-generating distribution on M' is taken to be the tangent bundle, $\xi' = TM'$ and the sub-Riemannian metric is a Riemannian metric. $\square$

We now define a notion of an embedding between metric spaces.

Definition 2.0.61. A Lipschitz map $\varphi : X \longrightarrow Y$ is *biLipschitz* if φ is injective and its inverse map

$$\varphi^{-1} : \varphi(X) \longrightarrow X$$

is also Lipschitz with respect to the metric d^Y restricted to $\varphi(X)$.

As with Lipschitz maps, we have a parallel notion of *locally biLipschitz* mappings.

Definition 2.0.62. A locally Lipschitz map $\varphi : X \longrightarrow Y$ is *locally biLipschitz* if φ is injective and its inverse map

$$\varphi^{-1} : \varphi(X) \longrightarrow X$$

is also locally Lipschitz with respect to the metric d^Y restricted to $\varphi(X)$.

BiLipschitz maps, for the purposes of this discussion, are the appropriate notion of equivalence between a metric space and its image. BiLipschitz embeddings will also be used to carry the notion of purely 2-unrectifiability from the first Heisenberg group $\mathbb{H}^1$ to any contact 3-manifold.

It will be of value to note that locally Lipschitz maps on compact sets are Lipschitz. In particular, we will use this to see that the Lipschitz homotopy groups we define in a later chapter agree with the Lipschitz homotopy groups that appear in [6].

Lemma 2.0.63. *Let $f : (X, d^X) \longrightarrow (Y, d^Y)$ be a locally Lipschitz map between metric spaces. If X is compact, then f is Lipschitz.*

Proof. Suppose that f is continuous but is not Lipschitz. Then, for every $n \in \mathbb{N}$, there exists $x_n, y_n \in X$ such that

$$d^Y(f(x_n), f(y_n)) > n \cdot d^X(x_n, y_n). \qquad (\diamond)$$

Since X is compact, $f(X)$ is a compact subset of the metric space Y, and thus is bounded. The sequence $(d^Y(x_n, y_n))$ has some upper bound $M \geq 0$ and, for all n,

$$M \geq d^Y(f(x_n), f(y_n)) > n \cdot d^X(x_n, y_n) \geq 0.$$

Therefore, the sequence $d^X(x_n, y_n)$ limits to 0 as n goes to infinity.

Note, since X is compact, through taking subsequences of (x_n) and (y_n), it can be assumed that (x_{n_k}) and (y_{n_k}) converge in X as n_k goes to infinity. Since $d^X(x_n, y_n)$ limits to 0, the subsequence both converge to the same point in X, denoted x.

It is now shown that f cannot be Lipschitz on any neighborhood of x, contradicting that f is locally Lipschitz. Let $\varepsilon > 0$ be fixed and consider $B(x, \varepsilon)$, the open ball centered at x of radius ε. Since f is locally Lipschitz, there exists Lipschitz constant $L_\varepsilon \geq 1$ for the map $f|_{B(x,\varepsilon)}$.

Since (x_{n_k}) and (y_{n_k}) both converge to x, there exists $N \in \mathbb{N}$ such that, for all $n_k \geq N$, elements of these sequences lie entirely in $B(x, \varepsilon)$. Furthermore, N make be taken to be larger than L_ε. In which case, by $(\diamond)$, for all $n_k \geq N$,

$$d^Y(f(x_{n_k}), f(y_{n_k})) > n_k \cdot d^X(x_{n_k}, y_{n_k}) > N \cdot d^X(x_{n_k}, y_{n_k}) > L_\varepsilon \cdot d^X(x_{n_k}, y_{n_k}).$$

But, as x_{n_k} and y_{n_k} are elements of $B(x, \varepsilon)$, this contradicts that f is L_ε-Lipschitz on this open ball. So, no Lipschitz constant exists on the open ball centered at x of radius $\varepsilon > 0$. Thus, since ε was arbitrary, there is no neighborhood about x on which f is Lipschitz. Thus, f is not locally Lipschitz, a contradiction.

□

Hausdorff Measure and Purely k-Unrectifiability

We now define some terms for reporting the metric structure of a metric space. In particular, a notion of volume of a subset of a metric space (Hausdorff measure) and a mapping in property which captures twisting of the space. In the following definitions, let (X, d^X) be a metric space.

Definition 2.0.64. The *diameter* of a non-empty, bounded set $A \subseteq X$, denoted $|A|$

or $diam_X(A)$, is defined by

$$|A| := \sup\{d^X(x,y) : x, y \in A\}.$$

Definition 2.0.65. For $\delta > 0$, a countable collection of sets $\{A_i\}_{i\in\mathbb{N}}$ in X is a δ-*cover* of a set $A \subseteq X$ if the collection forms a cover of A and each set has diameter at most δ, that is $A \subset \bigcup_{i=1}^{\infty} A_i$ and $0 \leq |A_i| < \delta$ for all i.

Fix a set $A \subseteq X$ and a non-negative integer k. For any $\delta > 0$, consider

$$\mathcal{H}^k_\delta(A) := \inf\left\{\sum_{i=1}^{\infty} |A_i|^k : \{A_i\} \text{ is a } \delta - \text{cover of } A\right\}.$$

For $0 < \delta' < \delta$, any δ'-cover of A is also a δ-cover of A and

$$\{\{A_i\} \text{ is a } \delta' - \text{cover of } A\} \subseteq \{\{A_i\} \text{ is a } \delta - \text{cover of } A\}.$$

As such,

$$\mathcal{H}^k_{\delta'}(A) \geq \mathcal{H}^k_\delta(A)$$

since restricting to a smaller collection yields a larger infimum. Since $\mathcal{H}^k_\delta(A)$ increases as δ decreases to 0, the limit as δ goes to 0 is some positive extended real number.

Definition 2.0.66. For a metric space (X, d^X) and a subset $A \subset X$, the k-*dimensional Hausdorff measure* of A is the positive extended real number

$$\mathcal{H}^k(A) := \lim_{\delta\to 0} \mathcal{H}^k_\delta(A).$$

The k-dimensional Hausdorff measure is a measure on the Borel σ-algebra on X as is verified in [11].

An advantage to considering the Hausdorff measure is that it allows for discussion of lower-dimensional volumes in higher dimensional spaces. For example, the image of a Lipschitz homotopy between paths in three-dimensional space will have measure zero with respect to the Lebesgue measure. Further, the image need not be an embedding or have other smooth properties, so defining and integrating a non-vanishing 2-form on the image could be impossible. But, the 2-dimensional Hausdorff measure of the image of the homotopy is defined and need not vanish.

The primary use in considering k-dimensional Hausdorff measure is in defining *purely k-unrectifiablility.*

Definition 2.0.67. A metric space (X, d) is *purely k-unrectifiable* for some integer $k \geq 1$ if, for all Borel sets $A \subset \mathbb{R}^k$ and all Lipschitz maps $f : A \longrightarrow X$, $\mathcal{H}^k(f(A)) = 0$.

In basic terms, a space is purely k-unrectifiable if Lipschitz maps from k-dimensional Euclidean space into the space cannot recover any of the Hausdorff measure. As will be shown, contact 3-manifolds are purely 2-unrectifiable. So, any Lipschitz homotopy between paths will have image of Hausdorff 2-measure zero. This essentially will say that Lipschitz homotopies between paths only occur if the paths bound no area.

Purely unrectifiable spaces are in contrast to rectifiable spaces, where the entirety of the measure of the space can be accounted for via the images of Lipschitz maps. For further discussion on these notions, see [1].

Hausdorff measure and Lipschitz maps

A useful characteristic of Hausdorff measure is that it is in some ways preserved by Lipschitz mappings. In particular and of great use to this dissertation, the image

of a $\mathcal{H}^k$-measure zero set under a Lipschitz map is $\mathcal{H}^k$-measure zero. The argument follows closely to the argument for Proposition 2.2 in [11].

Lemma 2.0.68. *Let* (X, d^X) *and* (Y, d^Y) *be metric spaces and let* $\varphi : X \longrightarrow Y$ *be a Lipschitz map. For* $k \geq 2$*, if* $\mathcal{H}^k(X) = 0$*, then* $\mathcal{H}^k(\varphi(X)) = 0$*.*

Proof. Take $L > 0$ to be the Lipschitz constant for φ. First, we will note that for any subset $A \subset X$, there is a bound of the diameter of the image of φ:

$$|\varphi(A)| \leq L\ |A|.$$

Indeed, for any $x, y \in A$, since φ is Lipschitz,

$$d^Y(\varphi(x), \varphi(y)) \leq L\ d^X(x, y) \leq L\ |A|.$$

So, $L\ |A|$ is an upper bound for $\{d^Y(\varphi(x), \varphi(y))\}_{x,y \in A}$. Thus, $|\varphi(A)| \leq L\ |A|$.

Now, let $\{A_i\}$ be a δ-cover of X for some $\delta > 0$. By the preceding argument and that $A_i \cap X \subset A_i$,

$$|\varphi(A_i \cap X)| \leq L\ |A_i \cap X| \leq L\ |A_i|$$

for each i. Thus, $\{\varphi(A_i \cap X)\}$ is an $L\delta$-cover of $\varphi(X)$. It follows that

$$\sum_{i=1}^{\infty} |\varphi(A_i \cap X)|^k \leq L^k \sum_{i=1}^{\infty} |A_i|^k$$

and $\mathcal{H}^k_{L\delta}(\varphi(X)) \leq L^k \mathcal{H}^k_\delta(X)$. Taking the limit as δ goes to 0 yields that

$$\mathcal{H}^k(\varphi(X)) \leq L^k \mathcal{H}^k(X).$$

If the space X is $\mathcal{H}^k$-measure zero, it follows immediately from this inequality that

$\varphi(X)$ is $\mathcal{H}^k$-measure zero too.

$\square$

In fact, $\mathcal{H}^k$-measure zero sets are preserved by locally Lipschitz maps as well.

Lemma 2.0.69. *Let (Y, d^Y) be a metric space, (X, d^X) be a second-countable metric space, and let $\varphi : X \longrightarrow Y$ be a locally Lipschitz map. For $k \geq 2$, if $\mathcal{H}^k(X) = 0$, then $\mathcal{H}^k(\varphi(X)) = 0$.*

Proof. Cover X by countably many sets $A_i \subset X$ such that $\varphi|_{A_i}$ is Lipschitz for each i. This is possible as X is second-countable. It follows by subadditivity of the measure $\mathcal{H}^k$ that $\mathcal{H}^k(A_i) = 0$ for each i.

By Lemma 2.0.68, $\mathcal{H}^k(\varphi(A_i)) = 0$. Since $\mathcal{H}^2$ is a measure on Y and $\varphi(X) = \cup_i \varphi(A_i)$, by sub-additivity of the measure,

$$0 \leq \mathcal{H}^k(\varphi(X)) \leq \sum_i \mathcal{H}^k(A_i) = 0.$$

$\square$

HOMOTOPY GROUPS OF SHEAVES

In this chapter, we define a notion of homotopy groups for pointed sheaves. The definition of homotopy groups of pointed sheaves is a generalization of homotopy groups of pointed topological spaces. That such a generalization exists is a folklore result that we make rigorous in this chapter. Homotopy groups of pointed sheaves will provide a framework to define horizontal homotopy groups and Lipschitz homotopy groups.

In this chapter, we will assume passing knowledge of category theory. For a general-purpose reference, see [2] or [21]. For a thorough rundown on sheaves, see [22].

Background

Sheaves

Definition 3.0.1. Let $\mathbf{C}$ be a category. Define the category of presheaves on $\mathbf{C}$ as

$$\mathbf{PShv}(\mathbf{C}) \coloneqq \mathbf{Fun}(\mathbf{C}^{op}, \mathbf{Set})$$

where the objects are functors from the opposite category $\mathbf{C}^{op}$ to the category of sets and morphisms are natural transformations between functors. An object in the category is called a *presheaf*.

Example 3.0.2. Let $\mathbf{C}$ be a category and $C \in \mathbf{C}$ be an object in said category. Then, define the **representable presheaf** of C to be a presheaf denoted $\hat{C}$ given by

$$\begin{aligned} \hat{C} : \mathbf{C}^{op} &\to \mathbf{Set} \\ A &\mapsto \{f : A \to C\} = \mathrm{Mor}_{\mathbf{C}}(A, C) \\ (A \xrightarrow{g} B) &\mapsto (f \overset{\hat{g}^*}{\mapsto} f \circ g) \end{aligned}$$

We will now make rigorous what it means for a presheaf to be a sheaf. Namely the presheaf must satisfy a descent condition with respect to open covers. Then, we will show that it is sufficient to check that the presheaf satisfies the descent condition for two-term open covers and for sequential open covers to establish that the presheaf is a sheaf. This fact is common folklore within the field of category theory. First, we will provide definitions for these types of covers.

Definition 3.0.3. For a manifold N, a *two-term open cover* is a pair of open subsets $\{U, V\}$ of N such that $N = U \cup V$.

Definition 3.0.4. For a manifold N, a *sequential open cover* is a collection of open subsets $\{U_i\}$ of N that are indexed by the natural numbers such that the open subsets are nested, $U_i \subset U_{i+1}$ for each i, and $N = \cup_{i=1}^{\infty} U_i$.

Definition 3.0.5. Let $\mathcal{M}$ be a category.

1. A *sieve (on $M \in \mathcal{M}$)* is a fully faithful right fibration $\mathcal{U} \subset \mathcal{M}_{/M}$ to the overcategory.

2. A *Grothendieck topology* on $\mathcal{M}$ is, for each object $M \in \mathcal{M}$, a subset $\mathsf{Cov}(M)$ of the set of sieves on M – elements of which are *covering sieves* – such that the following conditions are satisfied.

   (a) For each $M \in \mathcal{M}$, the overcategory $\mathcal{M}_{/M}$ is a covering sieve.

   (b) For each morphism $M \xrightarrow{f} M'$ in $\mathcal{M}$, and each covering sieve $\mathcal{U} \subset \mathcal{M}_{/M'}$, the sieve

   $$f^*\mathcal{U} := \mathcal{U} \underset{\mathcal{M}_{/M'}}{\times} \mathcal{M}_{/M} \subset \mathcal{M}_{/M}$$

   is a covering sieve.

(c) A sieve $\mathcal{U} \subset \mathcal{M}_{/M}$ is a covering sieve if there is a covering sieve $\mathcal{V} \subset \mathcal{M}_{/M}$ for which, for each object $(M' \xrightarrow{f} M) \in \mathcal{V}$, the sieve

$$f^*\mathcal{U} := \mathcal{U} \underset{\mathcal{M}_{/M}}{\times} \mathcal{M}_{/M'} \subset \mathcal{M}_{/M'}$$

is a covering sieve.

3. Fix a Grothendieck topology on $\mathcal{M}$. A presheaf $\mathcal{F} \in \mathbf{PShv}(\mathcal{M})$ is a *sheaf (on $\mathcal{M}$, with respect to its given Grothendieck topology)* if, for each covering sieve $\mathcal{U} \subset \mathcal{M}_{/M}$, the canonical map between sets

$$\mathcal{F}(M) \longrightarrow \mathsf{lim}\Big(\mathcal{U}^{\mathsf{op}} \hookrightarrow (\mathcal{M}_{/M})^{\mathsf{op}} \to \mathcal{M}^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathbf{Set}\Big)$$

is an isomorphism. The *category of sheaves (on $\mathcal{M}$, with respect to its given Grothendieck topology)* is the full subcategory

$$\mathbf{Shv}(\mathcal{M}) \subset \mathbf{PShv}(\mathcal{M})$$

consisting of the sheaves.

Example 3.0.6. The category **Man** of smooth manifolds and smooth maps thereamong, admits its *standard* Grothendieck topology: a sieve $\mathcal{U} \subset \mathbf{Man}_{/M}$ is a *covering sieve (on M)* if, for each point $p \in M$, there is an object $(U \xrightarrow{f} M) \in \mathcal{U}$ such that

- the smooth map f is an open embedding,
- the image $f(U) \ni p$ contains the point.

Convention 3.0.7. The category **Man** will be understood as equipped with its standard Grothendieck topology, thereby giving meaning to the notion of a *sheaf on*

Man.

Observation 3.0.8. Let M be a smooth manifold. Each collection $\mathcal{A}$ of open subsets of M determines a sieve:

$$\mathcal{U}_{\mathcal{A}} := \left\{ (U \xrightarrow{f} M) \in \mathbf{Man}_{/M} \;\middle|\; \begin{array}{c} \text{there is an element } A \in \mathcal{A} \\ \text{for which } f \text{ factors: } U \xrightarrow{f} A \hookrightarrow M \end{array} \right\} \subset \mathbf{Man}_{/M} .$$

The sieve $\mathcal{U}_{\mathcal{A}} \subset \mathbf{Man}_{/M}$ is a covering sieve if and only if $\mathcal{A}$ is an open cover of M.

Terminology 3.0.9. Let $\mathcal{F} \in \mathbf{PShv}(\mathbf{Man})$ be a presheaf.

- Let $\mathcal{A}$ be an open cover of a smooth manifold M. Say $\mathcal{F}$ *satisfies descent with respect to* $\mathcal{A}$ if the canonical map between sets

  $$\mathcal{F}(M) \longrightarrow \mathsf{lim}\Big(\mathcal{U}_{\mathcal{A}}^{\mathsf{op}} \to \mathbf{Man}_{/M}^{\mathsf{op}} \to \mathbf{Man}^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathbf{Set}\Big)$$

  is a bijection.

- Say $\mathcal{F}$ satisfies *two-term* descent if $\mathcal{F}$ satisfies descent with respect to each two-term open cover of each smooth manifold.

- Say $\mathcal{F}$ satisfies *finite* descent if $\mathcal{F}$ satisfies descent with respect to each finite open cover of each smooth manifold.

- Say $\mathcal{F}$ satisfies *sequential* descent if $\mathcal{F}$ satisfies descent with respect to each sequential open cover of each smooth manifold.

- Say $\mathcal{F}$ satisfies *countable* descent if $\mathcal{F}$ satisfies descent with respect to each countable open cover of each smooth manifold.

- Say $\mathcal{F}$ satisfies descent if $\mathcal{F}$ satisfies descent with respect to each open cover of each smooth manifold. [1]

Observation 3.0.10. A sieve $\mathcal{U} \subset \mathbf{Man}_{/M}$ is a covering sieve in the standard Grothendieck topology on **Man** if and only if there is an open cover $\mathcal{A}$ of M for which $\mathcal{U} = \mathcal{U}_{\mathcal{A}}$. Therefore, a presheaf $\mathcal{F} \in \mathbf{PShv}(\mathbf{Man})$ is a sheaf if and only if it satisfies descent.

Notation 3.0.11. Let $\mathcal{A}$ be a collection of open subsets of a smooth manifold M. Its *Cech* poset, $\mathfrak{C}(\mathcal{A})$, is that of non-empty finite subsets of $\mathcal{A}$, ordered by reverse inclusion: $R \leq S$ means $S \subset R$.

Lemma 3.0.12. *Let $\mathcal{A}$ be an open cover of a smooth manifold M. The functor*

$$\mathfrak{C}(\mathcal{A}) \longrightarrow \mathcal{U}_{\mathcal{A}} ~, \qquad R \mapsto \bigcap_{A \in R} A ~, \tag{3.1}$$

is final. In particular, for each functor $\mathcal{F}: \mathcal{U}_{\mathcal{A}}^{\mathsf{op}} \to \boldsymbol{Set}$, the canonical map between limit sets

$$\mathsf{lim}\Big(\mathcal{U}_{\mathcal{A}}^{\mathsf{op}} \xrightarrow{\mathcal{F}} \boldsymbol{Set}\Big) \longrightarrow \mathsf{lim}\Big(\mathfrak{C}(\mathcal{A})^{\mathsf{op}} \xrightarrow{(3.1)^{\mathsf{op}}} \mathcal{U}_{\mathcal{A}}^{\mathsf{op}} \xrightarrow{\mathcal{F}} \boldsymbol{Set}\Big)$$

is an isomorphism.

Proof. Through Quillen's Theorem A [29], it is enough to show, for each $(U \xrightarrow{f} M) \in \mathcal{U}_{\mathcal{A}}$, that the classifying space of the undercategory

$$\Big|\mathfrak{C}(\mathcal{A})^{(U \xrightarrow{f} M)/}\Big| \simeq *$$

is contractible. This undercategory $\mathfrak{C}(\mathcal{A})^{(U \xrightarrow{f} M)/}$ is the full subposet of $\mathfrak{C}(\mathcal{A})$ consisting of those non-empty finite subsets $R \in \mathcal{A}$ for which the image is contained

[1] Evidently, the condition that $\mathcal{F}$ satisfies descent is equivalent to the condition that $\mathcal{F}$ is a sheaf.

in the intersection:

$$f(U) \subset \bigcap_{A \in R} A \ .$$

By definition of $\mathcal{U}_{\mathcal{A}}$, this undercategory is non-empty. By direct inspection, this undercategory is closed under finite limits therein. Therefore, this undercategory is filtered. Contractibility of its classifying space follows.

□

Lemma 3.0.13. *Let $\mathcal{A}$ be an open cover of a smooth manifold M. Let $\mathcal{F} \in$ $\boldsymbol{PShv}(\boldsymbol{Man})$ be a presheaf. This presheaf $\mathcal{F}$ satisfies descent with respect to $\mathcal{A}$ if and only if the canonical map between sets*

$$\mathcal{F}(M) \longrightarrow \lim\Big(\mathfrak{C}(\mathcal{A})^{\mathsf{op}} \xrightarrow{(3.1)} \mathcal{U}_{\mathcal{A}}^{\mathsf{op}} \xrightarrow{\mathcal{F}} \boldsymbol{Set}\Big)$$

is an isomorphism.

Proof. Consider the canonical diagram among sets

$$\begin{tikzcd} \mathcal{F}(M) \arrow[rr] \arrow[drr] & & \lim\Big(\mathcal{U}_{\mathcal{A}}^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathbf{Set}\Big) \arrow[d] \\ & & \lim\Big(\mathfrak{C}(\mathcal{A})^{\mathsf{op}} \xrightarrow{(3.1)} \mathcal{U}_{\mathcal{A}}^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathbf{Set}\Big). \end{tikzcd}$$

Lemma 3.0.12 grants that the vertical map is an isomorphism. The result follows.

□

Observation 3.0.14. Let $\mathcal{A} = \{A_-, A_+\}$ be a two-term open cover of a smooth manifold M. Its Cech poset is the span:

$$\mathfrak{C}(\mathcal{A}) \ = \ \Big\{\{-\} \leftarrow \{-,+\} \rightarrow \{+\}\Big\} \ .$$

Therefore, Lemma 3.0.13 grants that a presheaf $\mathcal{F} \in \mathbf{PShv(Man)}$ satisfies descent with respect to $\mathcal{A}$ if and only if the canonical diagram among sets,

$$\begin{tikzcd}
\mathcal{F}(M) \arrow[r] \arrow[d] & \mathcal{F}(A_+) \arrow[d] \\
\mathcal{F}(A_-) \arrow[r] & \mathcal{F}(A_- \cap A_+),
\end{tikzcd}$$

is a pullback.

Lemma 3.0.15. *Let $\mathcal{A} = \{A_1 \subset A_2 \subset \dots\}$ be a sequential open cover of a smooth manifold M. A presheaf $\mathcal{F} \in \boldsymbol{PShv(Man)}$ satisfies descent with respect to $\mathcal{A}$ if and only if the canonical map between sets,*

$$\mathcal{F}(M) \longrightarrow \lim\Big(\dots \longrightarrow \mathcal{F}(A_{n+1}) \longrightarrow \mathcal{F}(A_n) \longrightarrow \dots \longrightarrow \mathcal{F}(A_1)\Big)$$

is an isomorphism.

Proof. Observe the canonical factorization:

$$\begin{tikzcd}
\mathfrak{C}(\mathcal{A})^{\mathsf{op}} \arrow[rr] \arrow[dr, "\mathsf{Min}^{\mathsf{op}}"'] & & \mathcal{U}_{\mathcal{A}}^{\mathsf{op}} \arrow[r, "\mathcal{F}"] & \mathbf{Set} \\
& (\mathbb{N}, \leq)^{\mathsf{op}} \arrow[ur, "n \mapsto A_n"'] & &
\end{tikzcd}$$

in where the functor Min is

$$\mathfrak{C}(\mathcal{A}) \ni R \mapsto \mathsf{Min}\Big\{n \in \mathbb{N} \mid A_n \in R\Big\} \in (\mathbb{N}, \leq) \ .$$

By direct inspection, this functor Min is locally Cartesian, and each fiber has a final object. It follows from Quillen's Theorem A [29] that the functor Min is final. After Lemma 3.0.12, the two-out-of-three property for final functors guarantees that the

functor

$$(\mathbb{N}, \leq) \xrightarrow{n \mapsto A_n} \mathcal{U}_{\mathcal{A}}$$

is final. The result follows.

□

Remark 3.0.16. Observation 3.0.14 and Lemma 3.0.15 together give a practical way to check two-term descent and sequential descent. The coming result, Proposition 3.0.19, in fact ensures that two-term descent and sequential descent imply descent.

Lemma 3.0.17. *Let $\mathcal{F} \in \boldsymbol{PShv}(\boldsymbol{Man})$ be a presheaf on the category of smooth manifolds and smooth maps thereamong. This presheaf $\mathcal{F}$ satisfies finite descent if and only if it satisfies two-term descent.*

Proof. Obviously, if $\mathcal{F}$ satisfies finite descent then it satisfies two-term descent. So it remains to prove the converse assertion.

Let $\mathcal{A}$ be a finite open cover of a smooth manifold M. Let $r = |\mathcal{A}|$ be the cardinality of $\mathcal{A}$. If $r = 1$, then $\mathcal{A}$ necessarily consists solely of the open subset $M \subset M$. In this case it is tautological that $\mathcal{F}$ satisfies descent with respect to $\mathcal{A}$.

If $r = 2$, then $\mathcal{F}$ satisfies descent with respect to $\mathcal{A}$ by assumption.

We proceed by induction on r, with $r = 1, 2$ as the base cases. So suppose $r > 2$. Let $A_0 \in \mathcal{A}$.

- Consider the open subset

$$M_0 := \bigcup_{A \in \mathcal{A} \smallsetminus \{A_0\}} A \subset M .$$

- Consider the open cover $\mathcal{A}_{\neq 0} := \mathcal{A} \smallsetminus \{A_0\}$ of M_0.

- Consider the open cover $\mathcal{A}_1 := \left\{A_0 \cap A \mid A \in \mathcal{A}_{\neq 0}\right\}$ of $A_0 \cap M_0$.

- Consider the open cover $\mathcal{B} := \left\{A_0, M_0\right\}$ of M.

- Consider the functor

$$\mathfrak{C}(\mathcal{A}) \longrightarrow \mathfrak{C}(\mathcal{B}) \ : \tag{3.2}$$

determined by the assignment

$$S \mapsto \begin{cases} \{A_0\} & , S = \{A_0\} \\ \{A_0, M_0\} & , A_0 \in S,\ S \neq \{A_0\} \\ \{M_0\} & , A_0 \notin S. \end{cases}$$

- Consider the right Kan extension

$$(3.2)_* \colon \mathfrak{C}(\mathcal{B})^{\mathsf{op}} \longrightarrow \mathbf{Set} \tag{3.3}$$

of the functor $\mathfrak{C}(\mathcal{A})^{\mathsf{op}} \to (\mathbf{Man}_{/M})^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathbf{Set}$ along (3.2).

By direct inspection, this functor (3.2) is a coCartesian fibration. Furthermore, the fiber over $\{A_0\}$ consists solely of $\{A_0\}$, while the fiber over $\{M_0\}$ is $\mathfrak{C}(\mathcal{A}_{\neq 0})$. Also, the fiber over $\{A_0, M_0\}$ is in bijective correspondence with $\mathfrak{C}(\mathcal{A}_1)$. Therefore, this right Kan extension (3.3) evaluates as limits of the fibers of (3.2), which we thusly identify:

- $$(3.2)_* \colon \{A_0\} \mapsto \mathcal{F}(A_0) \ .$$

- $$(3.2)_* \colon \{A_0, M_0\} \mapsto \lim\Big(\mathfrak{C}(\mathcal{A}_1)^{\mathsf{op}} \to (\mathbf{Man}_{/A_0 \cap M_0})^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathbf{Set}\Big) \ .$$

Because the cardinality $|\mathcal{A}_1| < r$ is strictly less than that of $\mathcal{A}$, the inductive hypothesis identifies this value of the right Kan extension as

$$(3.2)_*\colon \{A_0, M_0\} \mapsto \mathcal{F}(A_0 \cap M_0) \ .$$

- $$(3.2)_*\colon \{M_0\} \mapsto \lim\Big(\mathfrak{C}(\mathcal{A}_{\neq 0})^{\mathsf{op}} \to (\mathbf{Man}_{/M_0})^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathbf{Set}\Big) \ .$$

  Because the cardinality $|\mathcal{A}_{\neq 0}| < r$ is strictly less than that of $\mathcal{A}$, the inductive hypothesis identifies this value of the right Kan extension as

  $$(3.2)_*\colon \{M_0\} \mapsto \mathcal{F}(M_0) \ .$$

Through these identifications of the values of the right Kan extension $(3.2)_*$, we further identify this right Kan extension as the functor

$$(3.2)_*\colon \mathfrak{C}(\mathcal{B})^{\mathsf{op}} \longrightarrow (\mathbf{Man}_{/M})^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathbf{Set} \ , \qquad R \mapsto \mathcal{F}\Big(\bigcap_{A \in R} A\Big) \ .$$

Now, the canonical map under $\mathcal{F}(M)$ between limits

$$\lim\Big(\mathfrak{C}(\mathcal{B})^{\mathsf{op}} \to (\mathbf{Man}_{/M})^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathbf{Set}\Big) \xrightarrow{\cong} \lim\Big(\mathfrak{C}(\mathcal{A})^{\mathsf{op}} \to (\mathbf{Man}_{/M})^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathbf{Set}\Big)$$

is a bijection because right Kan extensions concatenate. Therefore, via Lemma 3.0.12, the result is proved upon showing that the canonical map

$$\mathcal{F}(M) \longrightarrow \lim\Big(\mathfrak{C}(\mathcal{B})^{\mathsf{op}} \to (\mathbf{Man}_{/M})^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathbf{Set}\Big)$$

is a bijection. Precisely because the cardinality $|\mathcal{B}| = 2$, our assumption on the presheaf

$\mathcal{F}$ is that this map is a bijection.

∎

Lemma 3.0.18. *Let $\mathcal{F} \in \boldsymbol{PShv}(\boldsymbol{Man})$ be a presheaf on the category $\boldsymbol{Man}$ of smooth manifolds and smooth maps thereamong. This presheaf $\mathcal{F}$ satisfies countable descent if and only if it satisfies two-term and sequential descent.*

Proof. By definition, two-term open covers and sequential open covers, are instances of countable open covers. Therefore the "only if" implication is manifestly true. Thusly, it remains to establish the "if" implication of the assertion. So suppose $\mathcal{F}$ satisfies descent with respect to two-term open covers and sequential open covers.

Let $\mathcal{A}$ be a countable open cover of a smooth manifold M. We must to show $\mathcal{F}$ satisfies descent with respect to $\mathcal{A}$. If $\mathcal{A}$ is finite, then Lemma 3.0.17 ensures that $\mathcal{F}$ satisfies descent with respect to $\mathcal{A}$. So assume $\mathcal{A}$ is infinite. Choose a bijection $\mathbb{N} \xrightarrow{n \mapsto A_n} \mathcal{A}$. We explain the canonical diagram among sets:

$$\begin{tikzcd}
\mathcal{F}(M) \arrow[rr] \arrow[dd] & & \lim\Big(\mathcal{U}_{\mathcal{A}}^{\mathsf{op}} \to \mathbf{Man}^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathbf{Set}\Big) \arrow[d] \\
& & \lim\Big(\mathfrak{C}(\mathcal{A})^{\mathsf{op}} \to \mathcal{U}_{\mathcal{A}}^{\mathsf{op}} \to \mathbf{Man}^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathbf{Set}\Big) \arrow[d] \\
\lim\Big((\mathbb{N},\leq)^{\mathsf{op}} \xrightarrow{n\mapsto A_n} \mathbf{Man}^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathbf{Set}\Big) \arrow[rr] & & \lim\Big((\mathbb{N},\leq)^{\mathsf{op}} \xrightarrow{\mathsf{Max}_*(\mathcal{F})} \mathbf{Set}\Big).
\end{tikzcd}$$

We must show the top horizontal map is an isomorphism. Lemma 3.0.12 states that the upper right vertical map is a bijection. Consider the functor between posets

$$\mathsf{Max}\colon \mathfrak{C}(\mathcal{A}) \longrightarrow (\mathbb{N},\leq) \ , \qquad R \mapsto \mathsf{Max}\big\{n \in \mathbb{N} \mid A_n \in R\big\} \ . \tag{3.4}$$

The functor $(\mathbb{N},\leq)^{\mathsf{op}} \xrightarrow{\mathsf{Max}_*(\mathcal{F})} \mathbf{Set}$ is the right Kan extension of $\mathcal{F}$ along (3.4).

Explicitly, this right Kan extension evaluates on $n \in \mathbb{N}$ as the limit indexed by the Cech poset of the open cover $\mathcal{A}_{\leq n} := \{A_k \mid k \leq n\}$ of the union $\bigcup\limits_{k \leq n} A_k$:

$$\mathsf{Max}_*(\mathcal{F})(n) \;=\; \mathsf{lim}\Big(\mathfrak{C}(\mathcal{A}_{\leq n})^{\mathsf{op}} = \mathfrak{C}(\mathbb{N}, \leq)^{\mathsf{op}}_{/n} \to \mathbf{Man}^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathbf{Set}\Big) \ ,$$

The lower right vertical map is an isomorphism because right Kan extensions concatenate. Because, for each $n \in \mathbb{N}$, the collection $\mathcal{A}_{\leq n}$ is a finite open cover of $\bigcup\limits_{k \leq n} A_k$, the lower horizontal map is an isomorphism because $\mathcal{F}$ is assumed to satisfy finite descent. The diagonal map is an isomorphism because $\mathcal{F}$ is assumed to satisfy sequential descent. The result follows from the two-out-of-three property for isomorphisms.

□

Proposition 3.0.19. *Let $\mathcal{F} \in$ **PShv**(**Man**) be a presheaf on the category **Man** of smooth manifolds and smooth maps thereamong. This presheaf $\mathcal{F}$ satisfies descent if and only if it satisfies two-term and sequential descent.*

Proof. Obviously, if $\mathcal{F}$ satisfies descent then it satisfies two-term and sequential descent. After Lemma 3.0.18, it remains to show that $\mathcal{F}$ satisfies descent if it satisfies countable descent.

So let $\mathcal{A}$ be an open cover of a smooth manifold M. Because M is second countable, there is a countable open cover $\mathcal{B}$ of M for which,

- for each $B \in \mathcal{B}$ there is an $A \in \mathcal{A}$ such that $B \subset A$;

- for each $A \in \mathcal{A}$ is the countable union:

$$\bigcup_{B \in \mathcal{B} \mid B \subset A} B \;=\; A \ .$$

In particular, $\mathcal{B}$ is a countable open cover of M, and for each $A \in \mathcal{A}$ the subset $\{B \in \mathcal{B} \mid B \subset A\} \subset \mathcal{B}$ is a countable open cover of A.

Consider the full subposet

$$\mathfrak{C}(\mathcal{B} \subset \mathcal{A}) \ := \ \Big\{(R, S) \in \mathfrak{C}(\mathcal{B}) \times \mathfrak{C}(\mathcal{A}) \mid \bigcap_{B \in R} B \subset \bigcap_{A \in S} A\Big\} \ \subset \ \mathfrak{C}(\mathcal{B}) \times \mathfrak{C}(\mathcal{A}) \ .$$

Projections from this poset fit into a lax-commutative diagram among posets

$$\begin{tikzcd}
\mathfrak{C}(\mathcal{B} \subset \mathcal{A}) \arrow[r, "\mathsf{pr}_{\mathcal{B}}"] \arrow[d, "\mathsf{pr}_{\mathcal{A}}"'] & \mathfrak{C}(\mathcal{B}) \arrow[r] & \mathcal{U}_{\mathcal{B}} \arrow[r] & \mathbf{Man}_{/M} \\
\mathfrak{C}(\mathcal{A}) \arrow[r] & \mathcal{U}_{\mathcal{A}} \arrow[urr, bend right] \arrow[r, phantom, "\Downarrow"] & {} & .
\end{tikzcd} \tag{3.5}$$

This diagram implements the sequence of maps among limit sets under the set $\mathcal{F}(M)$:

$$\begin{aligned}
\lim\Big(\mathcal{U}_{\mathcal{B}}^{\mathsf{op}} \xrightarrow{\mathcal{F}_{|\mathcal{U}_{\mathcal{B}}}} \mathbf{Set}\Big) \ &\xrightarrow{(a)} \ \lim\Big(\mathfrak{C}(\mathcal{B})^{\mathsf{op}} \xrightarrow{\mathcal{F}_{|\mathfrak{C}(\mathcal{B})}} \mathbf{Set}\Big) \\
&\xrightarrow{(b)} \ \lim\Big(\mathfrak{C}(\mathcal{B} \subset \mathcal{A})^{\mathsf{op}} \xrightarrow{\mathcal{F}_{|\mathfrak{C}(\mathcal{B} \subset \mathcal{A})}} \mathbf{Set}\Big) \\
&\xrightarrow{(c)} \ \lim\Big(\mathfrak{C}(\mathcal{A})^{\mathsf{op}} \xrightarrow{(\mathsf{pr}_{\mathcal{A}}^{\mathsf{op}})_* \mathcal{F}} \mathbf{Set}\Big) \\
&\xleftarrow{(d)} \ \lim\Big(\mathfrak{C}(\mathcal{A})^{\mathsf{op}} \xrightarrow{\mathcal{F}_{|\mathfrak{C}(\mathcal{A})}} \mathbf{Set}\Big) \\
&\xleftarrow{(f)} \ \lim\Big(\mathcal{U}_{\mathcal{A}}^{\mathsf{op}} \xrightarrow{\mathcal{F}_{|\mathcal{U}_{\mathcal{A}}}} \mathbf{Set}\Big).
\end{aligned}$$

We now verify that each of these maps is an isomorphism. Lemma 3.0.12 grants that the maps (a) and (f) are isomorphisms. The map (c) is an isomorphism because right Kan extensions concatenate.

To show (b) is an isomorphism it is enough to show the functor $\mathsf{pr}_{\mathcal{B}}$ is final. By Quillen's Theorem A [29], it is sufficient to show, for each object $R \in \mathfrak{C}(\mathcal{B})$, that the

undercategory has a contractible classifying space:

$$\left|\mathfrak{C}(\mathcal{B}\subset\mathcal{A})^{R/}\right| \;\simeq\; * \;.$$

This undercategory is the full subposet

$$\mathfrak{C}(\mathcal{B}\subset\mathcal{A})^{R/} \;=\; \{(R',S') \mid R\subset R'\} \;\subset\; \mathfrak{C}(\mathcal{B}\subset\mathcal{A}) \;.$$

By definition of $\mathcal{B}$, this undercategory is not empty. Also, by design, this undercategory admits finite limits. It follows that this undercategory is filtered, from which it follows further that this undercategory has contractible classifying space, as desired.

We next show that the map (d) is an isomorphism. For this, it is sufficient to show that the canonical natural transformation between functors $\mathfrak{C}(\mathcal{A})^{\mathsf{op}}\to\mathbf{Set}$,

$$\mathcal{F}_{|\mathfrak{C}(\mathcal{A})} \longrightarrow (\mathsf{pr}_{\mathcal{A}}^{\mathsf{op}})_*\mathcal{F} \tag{3.6}$$

is in fact a natural isomorphism. By direct inspection, the functor $\mathsf{pr}_{\mathcal{A}}$ is a coCartesian fibration. Consequently, the right Kan extension $(\mathsf{pr}_{\mathcal{A}}^{\mathsf{op}})\mathcal{F}$ evaluates as fiberwise limits. As so, the natural transformation (3.6) evaluates on an object $S\in\mathfrak{C}(\mathcal{A})$ as the canonical map between sets:

$$\mathcal{F}\Big(\bigcap_{A\in S}A\Big) \longrightarrow \mathsf{lim}\Big(\mathsf{pr}_{\mathcal{A}}^{-1}(S)^{\mathsf{op}} \xrightarrow{(R,S)\mapsto\mathcal{F}\left(\bigcap_{B\in R}B\right)} \mathbf{Set}\Big) \;.$$

Note that the fiber $\mathsf{pr}_{\mathcal{A}}^{-1}(S)=\mathfrak{C}(\mathcal{B}_S)$ is the Cech poset of the collection $\mathcal{B}_S := \Big\{B\in\mathcal{B} \mid B\subset \bigcap_{A\in S}A\Big\}$ of open subsets of the S-fold intersection of terms in $\mathcal{A}$. By construction of $\mathcal{B}$, this collection $\mathcal{B}_S$ is a countable open cover of this S-fold intersection, $\bigcap_{A\in S}A$.

The assumption that $\mathcal{F}$ satisfies countable descent exactly grants that the natural transformation (3.6) is indeed a natural isomorphism. We conclude that the map (d) is an isomorphism, as desired. This concludes the verification that each of the above maps is an isomorphism.

Because the above sequence of maps among sets lied canonically under the set $\mathcal{F}(M)$, the canonical map

$$\mathcal{F}(M) \longrightarrow \mathsf{lim}\Big(\mathcal{U}_{\mathcal{A}}^{\mathsf{op}} \xrightarrow{\mathcal{F}_{|\mathcal{U}_{\mathcal{A}}}} \mathbf{Set}\Big)$$

is an isomorphism if and only if the canonical map

$$\mathcal{F}(M) \longrightarrow \mathsf{lim}\Big(\mathcal{U}_{\mathcal{B}}^{\mathsf{op}} \xrightarrow{\mathcal{F}_{|\mathcal{U}_{\mathcal{B}}}} \mathbf{Set}\Big)$$

is an isomorphism. The latter is map is a bijection because $\mathcal{B}$ is a countable cover, and $\mathcal{F}$ is assumed to satisfy countable descent. We conclude that $\mathcal{F}$ satisfies descent with respect to $\mathcal{A}$, as desired.

$\square$

As we proceed, we will always use Proposition 3.0.19 to show that a given presheaf is a sheaf. The first instance of this strategy will be in Proposition 3.0.20 to show that the respresentable presheaf of a manifold is a sheaf.

Proposition 3.0.20. *Let N be a manifold. The representable presheaf*

$$\hat{N} : \boldsymbol{Man}^{op} \to \boldsymbol{Set}$$

is a sheaf.

Proof. Let $M \in \mathbf{Man}$ with two-term open cover $\{U, V\}$. Let $Z \in \mathbf{Set}$ such that the following diagram commutes:

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& \hat{N}(M) \arrow[r] \arrow[d] & \hat{N}(U) \arrow[d] \\
& \hat{N}(V) \arrow[r] & \hat{N}(U \cap V).
\end{tikzcd}$$

Then, for every $z \in Z$, $f(z)$ and $g(z)$ are smooth maps from U and V, respectively, to N that agree on $U \cap V$. We then define a map of sets $\varphi : Z \to \hat{N}(M)$ by

$$\varphi(z)(x) := \begin{cases} f(z)(x) & \text{if} \quad x \in U \\ g(z)(x) & \text{if} \quad x \in V. \end{cases}$$

This will be argued to be the unique filler of the diagram. Given another map $\varphi' : Z \to \hat{N}(M)$ filling the diagram, it must be that $\varphi'|_U = f$ and $\varphi'|_V = g$. Since the open sets U and V cover the manifold M, the map φ' agrees with the map φ everywhere on M. Thus, the presheaf $\hat{N}$ satisfies the descent condition for two-term open covers.

Let $\{U_i\}$ be a sequential open cover of the manifold M. Let $Z \in \mathbf{Set}$ such that the following solid diagram commutes:

$$\begin{tikzcd}
Z \arrow[d, dashed, "f"'] \arrow[drrr, bend left, "f_i"] \arrow[drrrrrr, bend left, "f_1"] & & & & & & \\
\hat{N}(M) \arrow[r] & \dots \arrow[r] & & \hat{N}(U_i) \arrow[r] & \dots \arrow[r] & & \hat{N}(U_1).
\end{tikzcd}$$

Define a map of sets $f : Z \longrightarrow \hat{N}(M)$ by

$$f(z)(x) := f_i(z)(x)$$

for any point $x \in M$, where the index i is such that $x \in U_i$. This map is well-defined since, for any pair of indices $j \geq i$ such that the point x is in U_i, the point x is in U_j and, as the diagram commutes, there is equality of points $f_j(z)(x) = f_i(z)(x)$. Also, since the open sets U_i cover the manifold M, the map f is defined on the entirety of M. Also, the map $f(z) : M \to N$ is smooth since it is smooth on each of the open sets U_i which cover M.

Finally, suppose that there exists another map $f' : Z \to \hat{N}(M)$ filling this diagram. Then, for each index i, there is equality of smooth maps from U_i to N:

$$f'(z)|_{U_i} = f_i(z) = f(z)|_{U_i}.$$

Since the open sets U_i cover the manifold M, the maps f and f' are equal. Thus, the presheaf $\hat{N}$ satisfies the descent condition for sequential open covers and, by Proposition 3.0.19, it is a sheaf. $\square$

Observation 3.0.21. Let N be a manifold. Consider the presheaf

$$\mathsf{Map}^{\mathsf{cont}}(-, N) : \mathbf{Man}^{op} \to \mathbf{Set}$$

that assigns to the manifold M the set of continuous maps $\mathsf{Map}^{\mathsf{cont}}(M, N)$ from the space M to the space N and assigns to smooth maps between manifolds the map given by precomposing by the smooth map. This presheaf closely resembles the representable presheaf $\hat{N}$ where smooth maps have been replaced by continuous maps in its definition. By a similar argument to the one provided in Proposition 3.0.20, the presheaf $\mathsf{Map}^{\mathsf{cont}}(-, N)$ is a sheaf.

Yoneda lemma

The following section covers fundamental tools in category theory, the Yoneda lemma and the Yoneda embedding. See [21] for proofs and further discussion.

Lemma 3.0.22. *(Yoneda Lemma) Let $\boldsymbol{C}$ be a category, $F : \boldsymbol{C}^{op} \to \boldsymbol{Set}$ be a functor, and $C \in \boldsymbol{C}$. Then there is a canonical bijection*

$$J : F(C) \to Nat(\hat{C}, F)$$

where $Nat(\hat{C}, F)$ is the set of natural transformations.

Next, there is a functor referred to as the Yoneda embedding. For a category $\mathbf{C}$, the following is a (fully-faithful) functor:

$$\begin{array}{rcl} j : \mathbf{C} & \to & \mathbf{PShv(C\ )} \\ C & \mapsto & \hat{C} \\ (C \xrightarrow{f} C') & \mapsto & \hat{f} \in \mathrm{Nat}(\hat{C}, \hat{C}') \end{array}$$

where $\hat{f}$ is the natural transformation between $\hat{C}$ and $\hat{C}'$ such that, for any $D \in \mathbf{C}$, $\hat{f}(D) : \hat{C}(D) \to \hat{C}'(D)$ is a map between sets defined by

$$\hat{f}(D)(D \xrightarrow{g} C) \coloneqq (D \xrightarrow{f \circ g} C').$$

Remark 3.0.23. The Yoneda embedding preserves products. In particular, for manifolds $M, N \in \mathbf{Man}$, the representable sheaf of the product of the manifolds is equivalent to the product of the representable sheafs:

$$\widehat{M \times N} = \widehat{M} \times \widehat{N}.$$

Observation 3.0.24. Let $\mathcal{F} : \mathbf{Man}^{op} \to \mathbf{Set}$ be a sheaf. Define a presheaf

$$\begin{array}{rcll} \mathrm{Nat}(\widehat{-},\mathcal{F}): & \mathbf{Man}^{op} & \to & \mathbf{Set} \\ & K & \mapsto & \mathrm{Nat}(\widehat{K},\mathcal{F}) \\ & (K \xrightarrow{f} K') & \mapsto & (\mathrm{Nat}(\widehat{K'},\mathcal{F}) \xrightarrow{\widehat{f}^*} \mathrm{Nat}(\widehat{K},\mathcal{F})). \end{array}$$

Via Yoneda Lemma, there is an isomorphism of functors $\mathrm{Nat}(\widehat{-},\mathcal{F}) \cong \mathcal{F}$. Thus, the presheaf $\mathrm{Nat}(\widehat{-},\mathcal{F})$ is a sheaf.

Pointed Natural Transformations

Definition 3.0.25. A *pointed sheaf* $(\mathcal{F}, P)$ is a pair consisting of a sheaf $\mathcal{F} \in \mathbf{Shv}(\mathbf{Man})$ and a natural transformation $P \in \mathrm{Nat}(\widehat{*},\mathcal{F})$.

By Yoneda lemma, the natural transformation P is associated to an element in the set $\mathcal{F}(*)$, which we will also refer to as P. If the sheaf $\mathcal{F}$ is representable, this is equivalent to selecting a base point in the associated manifold.

Definition 3.0.26. Define the category **ManSub** such that the objects are pairs (K, K_0) that consist of a manifold K and a submanifold $K_0 \subset K$. The morphisms between pairs (K, K_0) and (K', K_0') are smooth maps $f \in \mathsf{Map}^{\mathsf{sm}}(K, K')$ such that the image of the specified submanifold of the domain is contained in the specified submanifold of the target, that is, the following diagram commutes:

$$\begin{tikzcd} K \arrow[r, "f"] & K' \\ K_0 \arrow[u, hook, "incl_{K_0}"] \arrow[r, "f|_{K_0}"] & K_0'. \arrow[u, hook, "incl_{K_0'}"'] \end{tikzcd}$$

The composition rule is the composition of smooth maps.

Before we define the homotopy groups of a pointed sheaf, we introduce a construction that we will refer back to several times in defining pointed loops in sheaves and their homotopies.

Definition 3.0.27. Let $\mathbf{C}$ be a category and let $c_0, c, d \in \mathbf{C}$ be objects in the category. Let F and G be morphisms in the category $\mathbf{C}$ described diagrammatically below:

$$\begin{tikzcd} c_0 \arrow[r, "G"] \arrow[d, "F"'] & d. \\ c & \end{tikzcd}$$

Define $\mathrm{Mor}_G(c,d)$ via the following pullback diagram:

$$\begin{tikzcd} \mathrm{Mor}_G(c,d) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "\{G\}"] \\ \mathrm{Mor}(c,d) \arrow[r, "F^*"] & \mathrm{Mor}(c_0,d). \end{tikzcd}$$

An element φ in the set $\mathrm{Mor}_G(c,d)$ is a morphism $\varphi : c \longrightarrow d$ such that the following diagram commutes:

$$\begin{tikzcd} c_0 \arrow[r, "G"] \arrow[d, "F"'] & d. \\ c \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

Definition 3.0.28. For a given manifold and submanifold $(K, K_0) \in \mathbf{ManSub}$ and a pointed sheaf $(\mathcal{F}, P)$, define the set of *pointed natural transformations*

$$\mathrm{Nat}_P\left(\widehat{K}, \mathcal{F}\right)$$

by using Definition 3.0.27 with respect to the following diagram in the category $\mathbf{PShv(Man)}$:

$$\begin{tikzcd} \widehat{K_0} \arrow[r, "!"] \arrow[d, "\widehat{incl_{K_0}}"'] & \widehat{*} \arrow[r, "P"] & \mathcal{F}. \\ \widehat{K} & & \end{tikzcd}$$

For any pointed natural transformation $\varphi \in \mathrm{Nat}_P(\widehat{K}, \mathcal{F})$, the following is a commutative diagram:

$$\begin{tikzcd}
\widehat{K_0} \arrow[r, "!"] \arrow[d, "\widehat{incl_{K_0}}"'] & \widehat{*} \arrow[r, "P"] & \mathcal{F}. \\
\widehat{K} \arrow[urr, dashed, "\varphi"'] & &
\end{tikzcd}$$

In the case that the sheaf $\mathcal{F}$ is representable, this is equivalent to a smooth map φ being constant on the subspace K_0.

As will be shown, for any morphism $f : (K, K_0) \longrightarrow (K', K_0')$ in **ManSub**, we can pull back a pointed natural transformation from the representable sheaf $\widehat{K'}$ to the sheaf $\mathcal{F}$ by the morphism $\widehat{f}$ to get a pointed natural transformation from the representable sheaf $\widehat{K}$ to the sheaf $\mathcal{F}$ such that the chosen point P is unchanged.

Indeed, let $\varphi \in \mathrm{Nat}_P\left(\widehat{K'}, \mathcal{F}\right)$ be a pointed natural transformation and let $f : (K, K_0) \longrightarrow (K', K_0')$ be a smooth map sending the submanifold K_0 of K into the submanifold K_0' of K'.

Consider the pullback natural transformation $\widehat{f}^*(\varphi) : \widehat{K} \longrightarrow \mathcal{F}$. We will show that this natural transformation is pointed with respect to the submanifold K_0; that is, $\widehat{f}^*(\varphi) \circ \widehat{incl}_{K_0} = P$. Using the definition of pullbacks ,that the map f restricted to the submanifold K_0 maps into the submanifold K_0', and that the natural transformation φ is pointed, we have the following equality of natural transformations:

$$\begin{aligned}
\widehat{f}^*(\varphi) \circ \widehat{incl}_{K_0} &= \varphi \circ \widehat{f} \circ \widehat{incl}_{K_0} \\
&= \varphi \circ \widehat{incl}_{K_0'} \circ \widehat{f}|_{K_0} \\
&= P.
\end{aligned}$$

Thus, pullback by $\widehat{f}$ acts as desired and we can define the following presheaf on **ManSub**.

Definition 3.0.29. Let $(\mathcal{F}, P)$ be a pointed sheaf. Define a presheaf on the category

ManSub,

$$\mathrm{Nat}_P(\widehat{-},\mathcal{F}) : \mathbf{ManSub}^{op} \to \mathbf{Set}$$

as the assignments

$$\begin{array}{rcl} (K,K_0) & \mapsto & \mathrm{Nat}_P\left(\widehat{K},\mathcal{F}\right) \\ \left((K,K_0) \xrightarrow{f} (K',K_0')\right) & \mapsto & \left(\mathrm{Nat}_P\left(\widehat{K'},\mathcal{F}\right) \xrightarrow{\widehat{f}^*} \mathrm{Nat}_P\left(\widehat{K},\mathcal{F}\right)\right). \end{array}$$

Lemma 3.0.30. *Let $(\mathcal{F},P)$ be a pointed sheaf. The presheaf*

$$\mathit{Nat}_P(\widehat{-},\mathcal{F}) : \boldsymbol{ManSub}^{op} \to \boldsymbol{Set}$$

satisfies the descent condition for two-term open covers.

Proof. Let $K_0 \subset K$ be an object in **ManSub**. Consider a two-term open cover $\{U,V\}$ of the manifold K.

Then, the following pairs are objects in **ManSub**:

$$U \cap K_0 \subset U,\ V \cap K_0 \subset V,\ U \cap V \cap K_0 \subset U \cap V.$$

and the following diagram commutes:

$$\begin{tikzcd} (U\cap V, U\cap V\cap K_0) \arrow[r, "incl_{U\cap V}"] \arrow[d, "incl_{U\cap V}"'] & (U, U\cap K_0) \arrow[d, "incl_U"] \\ (V, V\cap K_0) \arrow[r, "incl_V"] & (K,K_0) \end{tikzcd}$$

Since $\mathrm{Nat}_P(\widehat{-},\mathcal{F})$ is a functor, there is a diagram induced by inclusions which is the back wall in the diagram below.

Let Z be a set with maps into the sets of pointed natural transformations $\mathrm{Nat}_P\left(\widehat{U},\mathcal{F}\right)$ and $\mathrm{Nat}_P\left(\widehat{V},\mathcal{F}\right)$ such that the following solid diagram commutes:

$$\begin{tikzcd}
Z \arrow[drrr, bend left] \arrow[dddr, bend right] \arrow[ddrr, dotted, "\varphi", "\exists !"'] & & & & \\
& \mathrm{Nat}_P(\widehat{K},\mathcal{F}) \arrow[rr] \arrow[dd] \arrow[dr, "\subset"'] & & \mathrm{Nat}_P(\widehat{U},\mathcal{F}) \arrow[dd] \arrow[dr, "\subset"] & \\
& & \mathrm{Nat}(\widehat{K},\mathcal{F}) \arrow[rr] \arrow[dd] & & \mathrm{Nat}(\widehat{U},\mathcal{F}) \arrow[dd] \\
& \mathrm{Nat}_P(\widehat{V},\mathcal{F}) \arrow[rr] \arrow[dr, "\subset"] & & \mathrm{Nat}_P(\widehat{U\cap V},\mathcal{F}) \arrow[dr, "\subset"] & \\
& & \mathrm{Nat}(\widehat{V},\mathcal{F}) \arrow[rr] & & \mathrm{Nat}(\widehat{U\cap V},\mathcal{F}).
\end{tikzcd}$$

Note that there is an natural inclusion of a set of pointed natural transformations into the set of natural transformation with the same domain and target functors. Since the presheaf $\mathrm{Nat}(\widehat{-},\mathcal{F})$ is a sheaf, there is a unique map of sets φ such that the diagram above commutes.

Take an element z in the set Z. Then, the natural transformation $\varphi(z) : \widehat{K} \to \mathcal{F}$ is such that there is an equality of natural transformations:

$$\varphi(z) \circ \widehat{incl_U} \circ \widehat{incl_{U\cap K_0}} = P \text{ and } \varphi(z) \circ \widehat{incl_V} \circ \widehat{incl_{V\cap K_0}} = P.$$

Since the open subsets $U \cap K_0$ and $V \cap K_0$ cover the submanifold K_0, these equivalences imply that $\varphi(z)$ is a pointed natural transformation:

$$\varphi(z) \circ \widehat{incl_{K_0}} = P$$

and that the map φ has image in the set $\mathrm{Nat}_P(\widehat{K},\mathcal{F})$.

The sheaf condition of the presheaf $\mathrm{Nat}\,(\widehat{-},\mathcal{F})$ guarantees that the map φ is the unique such map for which this diagram commutes. $\square$

Homotopy groups of pointed sheaves as a set

Primary definition of $\pi_n(\mathcal{F}, P)$

We aim to echo the definition of homotopy groups for manifolds in defining homotopy groups for sheaves. This is original work capturing folklore knowledge that such a notion of homotopy groups exists. If the sheaf $\mathcal{F} = \widehat{M}$ is representable for some $M \in \mathbf{Man}$, then the collection of smooth maps from $\mathbb{S}^n$ into M is isomorphic, via Yoneda lemma, to the natural transformations between the associated representable sheafs:

$$\mathrm{Nat}\left(\widehat{\mathbb{S}}^n, \mathcal{F}\right) \cong \mathsf{Map}^{\mathsf{sm}}(\mathbb{S}^n, M)$$

So, for a general sheaf, we consider the set of natural transformations $\mathrm{Nat}(\widehat{\mathbb{S}}^n, \mathcal{F})$ to act as the collection of maps from $\mathbb{S}^n$ into the sheaf $\mathcal{F}$.

We use Defintion 3.0.27 to define what a pointed map of $\mathbb{S}^n$ is for a pointed sheaf.

Convention 3.0.31. Throughout this chapter and in subsequent chapters, the base point of $\mathbb{S}^n$ is labeled as $0 \in \mathbb{S}^n$. Thus, the base point associated to the representable sheaf $\widehat{\mathbb{S}}^n$ is $\widehat{0}$.

Definition 3.0.32. For a pointed sheaf $(\mathcal{F}, P)$, define the set of pointed natural transformations $\mathrm{Nat}_P(\widehat{\mathbb{S}}^n, \mathcal{F})$ with reference to the following diagram in $\mathbf{PShv(Man)}$:

$$\begin{tikzcd} \widehat{*} \arrow[r, "P"] \arrow[d, "\widehat{0}"'] & \mathcal{F}. \\ \widehat{\mathbb{S}}^n & \end{tikzcd}$$

By construction, for any pointed natural transformation $\varphi \in \mathrm{Nat}_P(\widehat{\mathbb{S}}^n, \mathcal{F})$, there is the following commutative diagram:

$$\begin{tikzcd}
\widehat{*} \arrow[r, "P"] \arrow[d, "\widehat{0}"'] & \mathcal{F}. \\
\widehat{\mathbb{S}}^n \arrow[ur, dashed, "\varphi"'] &
\end{tikzcd}$$

The set $\mathrm{Nat}_P(\widehat{\mathbb{S}}^n, \mathcal{F})$ is the same set given by evaluating the sheaf $\mathrm{Nat}_P(\widehat{-}, \mathcal{F})$ on the pair $(\mathbb{S}^n, \{0\})$ in **ManSub**.

We also require an idea of homotopy between such pointed maps that fix the associated point P.

Definition 3.0.33. For a pointed sheaf $(\mathcal{F}, P)$, define the set of pointed natural transformations $\mathrm{Nat}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{S}}^n, \mathcal{F})$ with reference to the following diagram:

$$\begin{tikzcd}
\widehat{\mathbb{R}} \times \widehat{*} \arrow[r, "P"] \arrow[d, "(\mathbb{1}_{\mathbb{R}}, \widehat{0})"'] & \mathcal{F}. \\
\widehat{\mathbb{R}} \times \widehat{\mathbb{S}}^n &
\end{tikzcd}$$

By construction, for any pointed natural transformation $H \in \mathrm{Nat}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{S}}^n, \mathcal{F})$, there is the following commutative diagram:

$$\begin{tikzcd}
\widehat{\mathbb{R}} \times \widehat{*} \arrow[r, "P"] \arrow[d, "(\mathbb{1}_{\mathbb{R}}, \widehat{0})"'] & \mathcal{F}. \\
\widehat{\mathbb{R}} \times \widehat{\mathbb{S}}^n \arrow[ur, dashed, "H"'] &
\end{tikzcd}$$

The set $\mathrm{Nat}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{S}}^n, \mathcal{F})$ is equivalent to the sheaf $\mathrm{Nat}_P(\widehat{-}, \mathcal{F})$ evaluated on the pair $(\mathbb{R} \times \mathbb{S}^n, \mathbb{R} \times \{0\})$.

Definition 3.0.34. The *nth homotopy group* of a point sheaf $(\mathcal{F}, P)$ is the set

$$\pi_n(\mathcal{F}, P) \coloneqq \mathrm{Nat}_P(\widehat{\mathbb{S}}^n, \mathcal{F}) / \sim_n$$

where pointed natural transformations $\alpha, \beta \in \mathrm{Nat}_P(\widehat{\mathbb{S}}^n, \mathcal{F})$ are equivalent, $\alpha \sim_n \beta$, if there exists a pointed natural transformation $H \in \mathrm{Nat}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{S}}^n, \mathcal{F})$ such that the following diagram commutes:

$$\begin{tikzcd}
\widehat{\mathbb{S}^n} \arrow[d, "(\widehat{0}, \mathbb{1}_{\widehat{\mathbb{S}^n}})"'] \arrow[dr, "\alpha"] & \\
\widehat{\mathbb{R}} \times \widehat{\mathbb{S}^n} \arrow[r, "H"] & \mathcal{F}. \\
\widehat{\mathbb{S}^n} \arrow[u, "(\widehat{1}, \mathbb{1}_{\widehat{\mathbb{S}^n}})"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

If two natural transformations α and β are equivalent under the equivalence relation $\sim_n$, the natural transformations α and β are said to be *homotopic*. A natural transformation $H \in \mathrm{Nat}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{S}^n}, \mathcal{F})$ witnessing that α and β are homotopic is called a *homotopy* from α to β.

Before proceeding to the proof that the relation $\sim_n$ is an equivalence relation, we make note of some natural transformations and conventions that will be useful in the proof.

First, for any sheaf $\mathcal{G}$, there is a unique natural transformation to the sheaf $\widehat{*}$,

$$\mathcal{G} \xrightarrow{\widehat{!}} \widehat{*}.$$

We will use this fact often implicitly when discussing the point P of a pointed sheaf $(\mathcal{F}, P)$. When we speak of a natural transformation from a sheaf $\mathcal{G}$ to the sheaf $\mathcal{F}$ "equaling" the point P, we are meaning that the natural transformation is equal to the composition

$$\mathcal{G} \xrightarrow{\widehat{!}} \widehat{*} \xrightarrow{P} \mathcal{F}.$$

Let $F_{.5} \in \mathsf{Map}^{\mathsf{sm}}(\mathbb{R}, \mathbb{R})$ be the smooth map given by $F_{.5}(t) = 1 - t$ which flips 0 and 1. This determines a natural transformation from the sheaf $\widehat{\mathbb{R}}$ to itself: $\widehat{F_{.5}} \in \mathrm{Nat}(\widehat{\mathbb{R}}, \widehat{\mathbb{R}})$.

Let $\mathfrak{g}$ be a smooth self-map of $\mathbb{R}$ which restricts to the constant maps $\mathfrak{g}|_{(-\infty, \frac{1}{4}]} \equiv 0$ and $\mathfrak{g}|_{[\frac{3}{4}, \infty)} \equiv 1$ and which is strictly increasing on the interval $(1/4, 3/4)$. This also

determines a natural transformation from the sheaf $\widehat{\mathbb{R}}$ to itself: $\widehat{\mathfrak{g}} \in \mathrm{Nat}(\widehat{\mathbb{R}}, \widehat{\mathbb{R}})$.

Lemma 3.0.35. $\sim_n$ *is an equivalence relation on* $Nat_P(\widehat{\mathbb{S}}^n, \mathcal{F})$ *and thus* $\pi_n(\mathcal{F}, P)$ *is well-defined.*

Proof. First, to show reflexivity of $\sim_n$, note that there is a unique natural transformation

$$\widehat{\mathbb{R}} \xrightarrow{\widehat{!}} \widehat{*}.$$

Then, for any pointed natural transformation $\alpha \in \mathrm{Nat}_P\left(\widehat{\mathbb{S}}^n, \mathcal{F}\right)$, there is the natural transformation $\alpha \circ \left(\widehat{!}, \mathbb{1}_{\widehat{\mathbb{S}}^n}\right) \in \mathrm{Nat}_P\left(\widehat{\mathbb{R}} \times \widehat{\mathbb{S}}^n, \mathcal{F}\right)$ which is pointed as there is an equality of natural transformations

$$\alpha \circ \left(\widehat{!}, \mathbb{1}_{\widehat{\mathbb{S}}^n}\right) \circ \left(\mathbb{1}_{\widehat{\mathbb{R}}}, \widehat{0}\right) = \alpha \circ \left(\mathbb{1}_{\widehat{\mathbb{R}}}, \widehat{0}\right) = P.$$

The natural transformation $\alpha \circ (\widehat{!}, \mathbb{1}_{\widehat{\mathbb{S}}^n})$ is a homotopy from α to itself,

$$\begin{tikzcd}
\widehat{\mathbb{S}}^n \arrow[d, "(\widehat{0}{,}\mathbb{1}_{\widehat{\mathbb{S}}^n})"'] \arrow[drr, "\mathbb{1}_{\widehat{\mathbb{S}}^n}"] \arrow[drrr, bend left, "\alpha"] & & & \\
\widehat{\mathbb{R}} \times \widehat{\mathbb{S}}^n \arrow[rr, "(\widehat{!}{,}\mathbb{1}_{\widehat{\mathbb{S}}^n})"] & & \widehat{\mathbb{S}}^n \arrow[r, "\alpha"] & \mathcal{F}, \\
\widehat{\mathbb{S}}^n \arrow[u, "(\widehat{1}{,}\mathbb{1}_{\widehat{\mathbb{S}}^n})"] \arrow[urr, "\mathbb{1}_{\widehat{\mathbb{S}}^n}"'] \arrow[urrr, bend right, "\alpha"'] & & &
\end{tikzcd}$$

and the relation $\sim_n$ is refelxive.

For symmetry, suppose there is an equivalence of pointed natural transformations $\alpha \sim_n \beta$ which is witnessed by the homotopy $H \in \mathrm{Nat}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{S}}^n, \mathcal{F})$. Then, natural transformation

$$H \circ (\widehat{F_{.5}}, \mathbb{1}_{\widehat{\mathbb{S}}^n}) \in \mathrm{Nat}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{S}}^n, \mathcal{F})$$

is a homotopy from β to α:

$$\begin{tikzcd}
\widehat{\mathbb{S}^n} \arrow[d, "(\widehat{0},\mathbb{1}_{\widehat{\mathbb{S}^n}})"'] \arrow[dr, "(\widehat{1},\mathbb{1}_{\widehat{\mathbb{S}^n}})"] \arrow[drr, bend left, "\beta"] & & \\
\widehat{\mathbb{R}}\times\widehat{\mathbb{S}^n} \arrow[r, "(\widehat{F_{.5}},\mathbb{1}_{\widehat{\mathbb{S}^n}})"] & \widehat{\mathbb{R}}\times\widehat{\mathbb{S}^n} \arrow[r, "H"] & \mathcal{F}. \\
\widehat{\mathbb{S}^n} \arrow[u, "(\widehat{1},\mathbb{1}_{\widehat{\mathbb{S}^n}})"] \arrow[ur, "(\widehat{0},\mathbb{1}_{\widehat{\mathbb{S}^n}})"'] \arrow[urr, bend right, "\alpha"'] & &
\end{tikzcd}$$

The natural transformation is indeed pointed as there is an equality of natural transformations:

$$H \circ (\widehat{F_{.5}}, \mathbb{1}_{\widehat{\mathbb{S}^n}}) \circ (\mathbb{1}_{\widehat{\mathbb{R}}}, \widehat{0}) = H \circ (\mathbb{1}_{\widehat{\mathbb{R}}}, \widehat{0}) \circ (\widehat{F_{.5}}, \mathbb{1}_{\widehat{\mathbb{S}^n}}) = P.$$

Thus, $\beta \sim_n \alpha$ and the relation $\sim_n$ is symmetric.

Proceed to transitivity. Suppose that the pointed natural transformations $\alpha, \beta, \gamma \in \mathrm{Nat}_P(\widehat{\mathbb{S}^n}, \mathcal{F})$ are such that α is homotopic to β via a homotopy H and that β is homotopic to γ via a homotopy H'.

Define a two-term open cover of the space $\mathbb{R} \times \mathbb{S}^n$,

$$U = \left(-\infty, \frac{5}{8}\right) \times \mathbb{S}^n \text{ and } V = \left(\frac{3}{8}, \infty\right) \times \mathbb{S}^n.$$

In the following, let $(2t)$ and $(2t-1)$ denote smooth maps from $\mathbb{R}$ to itself defined by the procedure indicated within the notation. Via restriction and composition, we produce natural transformations

$$H \circ (\widehat{\mathfrak{g}}(2t), \mathbb{1}_{\widehat{\mathbb{S}^n}}) \in \mathrm{Nat}(\widehat{U}, \mathcal{F}) \text{ and } H' \circ (\widehat{\mathfrak{g}}(2t-1), \mathbb{1}_{\widehat{\mathbb{S}^n}}) \in \mathrm{Nat}(\widehat{V}, \mathcal{F}).$$

We now will observe that these natural transformations are pointed. For the submanifold $U \cap (\mathbb{R} \times \{0\}) = (-\infty, 5/8) \times \{0\}$ contained in $\mathbb{R} \times \mathbb{S}^n$, the natural

transformation $H \circ (\widehat{\mathfrak{g}}(2t), \mathbb{1}_{\widehat{\mathbb{S}}^n})$ is pointed:

$$H \circ (\widehat{\mathfrak{g}}(2t), \mathbb{1}_{\widehat{\mathbb{S}}^n}) \circ \widehat{incl}_{(-\infty,5/8)\times\{0\}} = H \circ (\widehat{\mathfrak{g}}(2t), \widehat{0}) = P.$$

Similarly, for the submanifold $V \cap (\mathbb{R} \times \{0\}) = (3/8, \infty) \times \{0\}$ contained in $\mathbb{R} \times \mathbb{S}^n$, the natural transformation $H' \circ (\widehat{\mathfrak{g}}(2t-1), \mathbb{1}_{\widehat{\mathbb{S}}^n})$ is pointed:

$$H' \circ (\widehat{\mathfrak{g}}(2t-1), \mathbb{1}_{\widehat{\mathbb{S}}^n}) \circ \widehat{incl}_{(3/8,\infty)\times\{0\}} = H' \circ (\widehat{\mathfrak{g}}(2t-1), \widehat{0}) = P.$$

Now, on the submanifold $U \cap V \cap (\mathbb{R} \times \mathbb{S}^n) = (3/8, 5/8) \times \mathbb{S}^n$ in $\mathbb{R} \times \mathbb{S}^n$, the two pointed natural transformations agree:

$$H \circ (\widehat{\mathfrak{g}}(2t), \mathbb{1}_{\widehat{\mathbb{S}}^n}) = H \circ (\widehat{1}, \mathbb{1}_{\widehat{\mathbb{S}}^n}) = \beta = H' \circ (\widehat{0}, \mathbb{1}_{\widehat{\mathbb{S}}^n}) = H' \circ (\widehat{\mathfrak{g}}(2t-1), \mathbb{1}_{\widehat{\mathbb{S}}^n}).$$

So, the following diagram commutes and, since the presheaf $\mathrm{Nat}_P(\widehat{-}, \mathcal{F})$ satisfies the descent condition for two-term open covers (Lemma 3.0.30), there exists a unique filler $H * H'$:

$$\begin{tikzcd}
* \arrow[dr, dotted, "H*H'", "\exists!"'] \arrow[drr, bend left, "H\circ(\widehat{\mathfrak{g}}(2t),\mathbb{1}_{\widehat{\mathbb{S}}^n})"] \arrow[ddr, bend right, "H'\circ(\widehat{\mathfrak{g}}(2t-1),\mathbb{1}_{\widehat{\mathbb{S}}^n})"'] & & \\
& \mathrm{Nat}_P(\widehat{\mathbb{R}}\times\widehat{\mathbb{S}}^n,\mathcal{F}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Nat}_P(\widehat{U},\mathcal{F}) \arrow[d] \\
& \mathrm{Nat}_P(\widehat{V},\mathcal{F}) \arrow[r] & \mathrm{Nat}_P(\widehat{U\cap V},\mathcal{F}).
\end{tikzcd}$$

The pointed natural transformation $H * H'$ then satisfies the following diagram and $\alpha \sim_n \gamma$:

$$\begin{tikzcd}
\widehat{\mathbb{S}^n} \arrow[d, "(\widehat{0},\mathbb{1}_{\widehat{\mathbb{S}^n}})"'] \arrow[drr, bend left, "H\circ(\widehat{\mathfrak{g}}(0),\mathbb{1}_{\widehat{\mathbb{S}^n}})=H\circ(\widehat{0},\mathbb{1}_{\widehat{\mathbb{S}^n}})=\alpha"] & & \\
\widehat{\mathbb{R}}\times\widehat{\mathbb{S}^n} \arrow[rr, "H*H'"] & & \mathcal{F}. \\
\widehat{\mathbb{S}^n} \arrow[u, "(\widehat{1},\mathbb{1}_{\widehat{\mathbb{S}^n}})"] \arrow[urr, bend right, "H'\circ(\widehat{\mathfrak{g}}(1),\mathbb{1}_{\widehat{\mathbb{S}^n}})=H'\circ(\widehat{1},\mathbb{1}_{\widehat{\mathbb{S}^n}})=\gamma"'] & &
\end{tikzcd}$$

Therefore, the relation $\sim_n$ is transitive and an equivalence relation. □

The next propositions affirms that the homotopy groups agree between a manifold and its representable sheaf. Note that, via the Whitney approximation theorem, the continuous version of homotopy groups of a manifold can be replaced with the smooth version of the definition, where in all maps of the n-sphere are taken to be smooth as are the homotopies between such maps.

Lemma 3.0.36. *For any manifold $M \in$ **Man** and any point $p \in M$, there is a canonical bijection of sets*

$$\pi_n(M,p) \cong \pi_n\left(\widehat{M},\widehat{p}\right).$$

Proof. The set of smooth maps from $\mathbb{S}^n$ into M based at the point p is denoted $\mathsf{Map}_p^{\mathsf{sm}}(\mathbb{S}^n, M)$ and is defined using Definition 3.0.27 with respect to the following diagram of constant smooth maps in **Man**:

$$\begin{tikzcd}
* \arrow[r, "p"] \arrow[d, "0"'] & M. \\
\mathbb{S}^n &
\end{tikzcd}$$

Via the Yoneda embedding, there are canonical bijections of sets

$$\mathsf{Map}^{\mathsf{sm}}(\mathbb{S}^n, M) \cong \mathrm{Nat}(\widehat{\mathbb{S}^n}, \widehat{M}) \text{ and } \mathsf{Map}^{\mathsf{sm}}(*, M) \cong \mathrm{Nat}(\widehat{*}, \widehat{M}).$$

As the set of pointed smooth maps $\mathsf{Map}_p^{\mathsf{sm}}(\mathbb{S}^n, M)$ and the set of pointed natural transformations $\mathrm{Nat}_{\widehat{p}}(\widehat{\mathbb{S}^n}, \widehat{M})$ are both defined via pullback of bijective sets, they too

are in bijection:

$$
\begin{tikzcd}
\mathsf{Map}^{\mathsf{sm}}_p(\mathbb{S}^n, M) \arrow[rr] \arrow[dd] \arrow[dr, dotted, "j", "\exists\,\cong"'] \arrow[dr, phantom, very near start, "\lrcorner"] & & * \arrow[dr, "\cong"] \arrow[dd, "\{p\}"' near start] & \\
& \mathrm{Nat}_{\widehat{p}}(\widehat{\mathbb{S}}^n, \widehat{M}) \arrow[rr, crossing over] \arrow[dr, phantom, very near start, "\lrcorner"] & & * \arrow[dd, "\{\widehat{p}\}"] \\
\mathsf{Map}^{\mathsf{sm}}(\mathbb{S}^n, M) \arrow[rr, "0^*" near end] \arrow[dr, "\cong"] & & \mathsf{Map}^{\mathsf{sm}}(*, M) \arrow[dr, "\cong"] & \\
& \mathrm{Nat}(\widehat{\mathbb{S}}^n, \widehat{M}) \arrow[rr, "\widehat{0}^*"] \arrow[from=uu, crossing over] & & \mathrm{Nat}(\widehat{*}, \widehat{M}).
\end{tikzcd}
$$

By a similar argument, there is a bijection

$$j : \mathsf{Map}^{\mathsf{sm}}_p(\mathbb{R} \times \mathbb{S}^n, M) \cong \mathrm{Nat}_{\widehat{p}}(\widehat{\mathbb{R}} \times \widehat{\mathbb{S}}^n, \widehat{M})$$

and the following diagram commutes:

$$
\begin{tikzcd}
\mathsf{Map}^{\mathsf{sm}}_p(\mathbb{R} \times \mathbb{S}^n, M) \arrow[r, "j", "\cong"'] \arrow[d, "ev_{0,1}"] & \mathrm{Nat}_{\widehat{p}}(\widehat{\mathbb{R}} \times \widehat{\mathbb{S}}^n, \widehat{M}) \arrow[d, "ev_{\widehat{0},\widehat{1}}"] \\
\mathsf{Map}^{\mathsf{sm}}_p(\mathbb{S}^n, M) \times \mathsf{Map}^{\mathsf{sm}}_p(\mathbb{S}^n, M) \arrow[r, "j", "\cong"'] & \mathrm{Nat}_{\widehat{p}}(\widehat{\mathbb{S}}^n, \widehat{M}) \times \mathrm{Nat}_{\widehat{p}}(\widehat{\mathbb{S}}^n, \widehat{M}).
\end{tikzcd}
\tag{3.7}
$$

Here, the maps $ev_{0,1}$ and $ev_{\widehat{0},\widehat{1}}$ are evaluation maps:

$$
\begin{array}{rccl}
ev_{0,1} : & \mathsf{Map}^{\mathsf{sm}}_p(\mathbb{R} \times \mathbb{S}^n, M) & \to & \mathsf{Map}^{\mathsf{sm}}_p(\mathbb{S}^n, M) \times \mathsf{Map}^{\mathsf{sm}}_p(\mathbb{S}^n, M) \\
& H & \mapsto & (H \circ (0, \mathbb{1}_{S^n}), H \circ (1, \mathbb{1}_{S^n})).
\end{array}
$$

$$
\begin{array}{rccl}
ev_{\widehat{0},\widehat{1}} : & \mathrm{Nat}_{\widehat{p}}(\widehat{\mathbb{R}} \times \widehat{\mathbb{S}}^n, \widehat{M}) & \to & \mathrm{Nat}_{\widehat{p}}(\widehat{\mathbb{S}}^n, \widehat{M}) \times \mathrm{Nat}_{\widehat{p}}(\widehat{\mathbb{S}}^n, \widehat{M}) \\
& H & \mapsto & (H \circ (\widehat{0}, \mathbb{1}_{\widehat{S}^n}), H \circ (\widehat{1}, \mathbb{1}_{\widehat{S}^n})).
\end{array}
$$

We now define a map between the sets $\pi_n(M, p)$ and $\pi_n(\widehat{M}, \widehat{p})$:

$$\begin{array}{rccc} j: & \pi_n(M,p) & \to & \pi_n\left(\widehat{M},\widehat{p}\right) \\ & [\ \alpha\ ] & \mapsto & [\ \widehat{\alpha}\ ]. \end{array}$$

We first argue that this map is well-defined.

Two pointed maps $\alpha, \beta \in \mathsf{Map}_p^{\mathsf{sm}}(\mathbb{S}^n, M)$ are equivalent in the nth homotopy group $\pi_n(M,p)$ if there exists a smooth homotopy $H \in \mathsf{Map}_p^{\mathsf{sm}}(\mathbb{R} \times \mathbb{S}^n, M)$ such that $ev_{0,1}(H) = (\alpha, \beta)$. Since diagram (3.7) commutes, the image of the homotopy H under the bijection j is a pointed natural transformation $jH = \widehat{H} \in \mathrm{Nat}_{\widehat{p}}(\widehat{\mathbb{R}} \times \widehat{\mathbb{S}}^n, \widehat{M})$ and we have an equality of pairs of pointed natural transformations:

$$ev_{\widehat{0},\widehat{1}}(\widehat{H}) = (\widehat{H} \circ (\widehat{0}, \mathbb{1}_{\widehat{S}^n}), \widehat{H} \circ (\widehat{1}, \mathbb{1}_{\widehat{S}^n})) = (j\alpha, j\beta) = (\widehat{\alpha}, \widehat{\beta}).$$

So, the pointed natural transformations $\widehat{\alpha}$ and $\widehat{\beta}$ are homotopic via the pointed natural transformation $\widehat{H}$, and the map j is well-defined.

Repeating the same process with the bijection j^{-1} yields the inverse map to the map j. Thus, there is a bijection between these two notions of homotopy group when the sheaf is representable.

□

Alternate definition of $\pi_n(\mathcal{F}, P)$

Before assembling the group structure on $\pi_n(\mathcal{F}, P)$, it will be convenient to introduce one more definition of homotopy groups of a sheaf which will make the means of concatenating loops more palatable. In the next section, it will be shown that the two notions of homotopy groups are equivalent.

Going forward, let I denote the closed interval $[0,1]$.

Definition 3.0.37. For manifold and submanifold pair $(K, K_0) \in$ **ManSub**, consider pointed natural transformations $\varphi \in \mathrm{Nat}_P(\widehat{K}, \mathcal{F})$ such that there exists an open

set $W_\varphi \subset K$ that contains the closure of K_0 and such that the following diagram commutes:

$$\begin{tikzcd} \widehat{K_0} \arrow[r, "\widehat{incl_{K_0}}"] \arrow[dr, "\widehat{incl_{K_0}}"'] & \widehat{U}_\varphi \arrow[r, "!"] \arrow[d, "\widehat{incl}_{W_\varphi}"] & \widehat{*} \arrow[d, "P"] \\ & \widehat{K} \arrow[r, "\varphi"] & \mathcal{F}. \end{tikzcd}$$

Denote the set of all such natural transformations as $\widetilde{\mathrm{Nat}}_P(\widehat{K}, \mathcal{F})$.

Definition 3.0.38. Let the set $\widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$ refer to Definition 3.0.37 with respect to the manifold-submanifold pair $(\mathbb{R}^n, \mathbb{R}^n \smallsetminus I^n)$.

Observation 3.0.39. For any natural transformation $\varphi \in \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$, the natural transformation is constant outside of the closed box I^n in the following sense:

$$\begin{tikzcd} \widehat{\mathbb{R}^n \smallsetminus I^n} \arrow[r, "\widehat{incl}_{\mathbb{R}^n \smallsetminus I^n}"] \arrow[dr, "\widehat{incl}_{\mathbb{R}^n \smallsetminus I^n}"'] & \widehat{U}_\varphi \arrow[r, "!"] \arrow[d, "\widehat{incl}_{W_\varphi}"] & \widehat{*} \arrow[d, "P"] \\ & \widehat{\mathbb{R}}^n \arrow[r, "\varphi"] & \mathcal{F}. \end{tikzcd}$$

Definition 3.0.40. Let the set $\widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n, \mathcal{F})$ refer to Definition 3.0.37 with respect to the manifold-submanifold pair $(\mathbb{R} \times \mathbb{R}^n, \mathbb{R} \times (\mathbb{R}^n \smallsetminus I^n))$.

Definition 3.0.41. The *nth homotopy group* of a point sheaf $(\mathcal{F}, P)$ is the set

$$\pi_n(\mathcal{F}, P) \coloneqq \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F}) / \sim_{n'}$$

where pointed natural transformations $\alpha, \beta \in \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$ are equivalent, $\alpha \sim_{n'} \beta$, if there exists a pointed natural transformation $H \in \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n, \mathcal{F})$ such that the following diagram commutes:

$$\begin{tikzcd} \widehat{\mathbb{R}}^n \arrow[d, "(\widehat{0}, \mathbb{1}_{\widehat{\mathbb{R}}^n})"'] \arrow[dr, "\alpha"] & \\ \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n \arrow[r, "H"] & \mathcal{F}. \\ \widehat{\mathbb{R}}^n \arrow[u, "(\widehat{1}, \mathbb{1}_{\widehat{\mathbb{R}}^n})"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

If two natural transformations α and β are equivalent under the equivalence relation $\sim_{n'}$, the natural transformations α and β are said to be *homotopic.* A natural transformation $H \in \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n, \mathcal{F})$ witnessing that α and β are homotopic is called a *homotopy* from α to β.

Observation 3.0.42. By an argument similar to the one used to prove Lemma 3.0.35, the relation $\sim_{n'}$ is an equivalence relation and this definition of the nth homotopy group of a pointed sheaf is well-defined.

The use of the same terms and notation for this definition of homotopy groups and for Definition 3.0.33 is justified as these definitions are equivalent, which will be shown in the next section.

Equivalence of definitions of $\pi_n(\mathcal{F}, P)$

Lemma 3.0.43. *The definitions of the nth homotopy group presented in Definition 3.0.33 and Definition 3.0.41 are equivalent. That is, there exists an isomorphism between the sets*

$$Nat_P(\widehat{\mathbb{S}}^n, \mathcal{F}) / \sim_n \xrightarrow{\cong} \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F}) / \sim_{n'} .$$

Proof. Let $\psi : \mathbb{R}^n \to \mathbb{S}^n$ be a smooth map such that there exists an open subset $U \subset \mathbb{R}^n$ such that

- the open subset U contains the closed subset $\overline{\mathbb{R}^n \setminus I^n}$,
- the map ψ maps only the closure of the open subset U to 0; $\psi(\overline{U}) = \{0\}$, and
- the restriction map $\psi|_{\mathbb{R}^n \setminus \overline{U}}$ is a diffeomorphism onto the n-sphere minus its base point, $\mathbb{S}^n \setminus \{0\}$.

Such a map can be thought of as stretching the entirety of $\mathbb{R}^n$ around $\mathbb{S}^n$ and smoothly pinching the excess Euclidean space of $\overline{U}$ to 0.

Define a mapping

$$\begin{array}{rcll} \widehat{\psi}^{*}: & \mathrm{Nat}_P(\widehat{\mathbb{S}}^n, \mathcal{F})/\sim_n & \to & \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})/\sim_{n'} \\ & [\ \alpha\ ] & \mapsto & [\ \alpha \circ \widehat{\psi}\ ]. \end{array}$$

First, we will note that precomposing elements of the set $\mathrm{Nat}_P(\widehat{\mathbb{S}}^n, \mathcal{F})$ by the map $\widehat{\psi}$ yields elements of the set $\widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$. Take a pointed natural transformation $\alpha \in \mathrm{Nat}_P(\widehat{\mathbb{S}}^n, \mathcal{F})$ on $\mathbb{S}^n$. Since the map ψ sends the open set U to the base point of $\mathbb{S}^n$, there is an equivalence of natural transformations,

$$\alpha \circ \widehat{\psi} \circ \widehat{incl}_U = \alpha \circ \widehat{0} = P.$$

Then, $\alpha \circ \widehat{\psi}$ is a pointed natural transformation in $\widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$.

We now show that the map $\widehat{\psi}^{*}$ is well-defined. Suppose that pointed natural transformations $\alpha, \alpha' \in \mathrm{Nat}_P(\widehat{\mathbb{S}}^n, \mathcal{F})$ were equivalent with respect to $\sim_n$ via a pointed natural transformation $H \in \mathrm{Nat}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{S}}^n, \mathcal{F})$. Precomposing the natural transformation H by the natural transformation $(\mathbb{1}_{\widehat{\mathbb{R}}}, \widehat{\psi})$ yields a natural transformation $H \circ (\mathbb{1}_{\mathbb{R}}, \widehat{\psi}) \in \mathrm{Nat}(\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n, \mathcal{F})$.

In fact, this natural transformation will yield a homotopy in the $\sim_{n'}$-sense. Since the smooth mapping ψ sends the open set U to the base point in $\mathbb{S}^n$,

$$H \circ (\mathbb{1}_{\mathbb{R}}, \widehat{\psi}) \circ \widehat{incl}_{\mathbb{R}\times U} = H \circ (\mathbb{1}_{\mathbb{R}}, \widehat{0}) = P$$

and $H \circ (\mathbb{1}_{\mathbb{R}}, \widehat{\psi}) \in \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n, \mathcal{F})$.

Then, this natural transformation yields a homotopy in the $\sim_{n'}$-sense between natural transformations $H \circ (\widehat{0}, \widehat{\psi}) = \alpha \circ \widehat{\psi}$ and $H \circ (\widehat{1}, \widehat{\psi}) = \alpha' \circ \widehat{\psi}$. So, there is a $\sim_{n'}$-equivalence $\alpha \circ \widehat{\psi} \sim_{n'} \alpha' \circ \widehat{\psi}$ and the map $\widehat{\psi}^{*}$ is well-defined.

We now will show surjectivity of the map $\widehat{\psi}^{*}$. Let $[\ \beta\ ] \in \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})/\sim_{n'}$.

A representative β can be found from this equivalence class via scaling such that the open set W_β on which the pointed natural transformation β is constant properly contains the closed set $\overline{U}$.

Consider the open set $W_\beta \smallsetminus \overline{U}$. Since the restriction map $\psi|_{\mathbb{R}^n \smallsetminus \overline{U}}$ is a differomorphism, the set $B := \psi|_{\mathbb{R}^n \smallsetminus \overline{U}}(W_\beta \smallsetminus \overline{U})$ is an open set in $\mathbb{S}^n \smallsetminus \{0\}$. Since $\overline{U} \subset W_\beta$ is a proper containment, there is an equality of sets in $\mathbb{S}^n$,

$$\psi(W_\beta) = \psi|_{\mathbb{R}^n \smallsetminus \overline{U}}(W_\beta \smallsetminus \overline{U}) \cup \psi(\overline{U}) = B \cup \{0\}$$

and the set $\psi(W_\beta)$ is an open neighborhood of the base point $0 \in \mathbb{S}^n$.

So, there is a natural transformation

$$\eta := \beta \circ \widehat{\psi}|^{-1}_{\mathbb{R}^n \smallsetminus \overline{U}} : \widehat{\mathbb{S}^n \smallsetminus \{0\}} \longrightarrow \mathcal{F}$$

which satisfies the following equality of natural transformations,

$$\eta \circ \widehat{incl_B} = \beta \circ \widehat{\psi}^{-1}|_{\mathbb{R}^n \smallsetminus \overline{U}} \circ \widehat{incl}_B = \beta \circ \widehat{incl_{W_\beta}} \circ \widehat{\psi}^{-1}|_{\mathbb{R}^n \smallsetminus \overline{U}} \circ \widehat{incl_B} = P.$$

Since $\psi(W_\beta) = B \cup \{0\}$ is an open neighborhood of the base point 0 in $\mathbb{S}^n$ and the natural transformation η is constant on $\psi(W_\beta) \smallsetminus \{0\} = B$, the natural transformation can be extended to all of $\widehat{S}^n$ such that

$$\eta \circ \widehat{0} = P.$$

Thus, we have a pointed natural transformation $\eta \in \mathrm{Nat}_P(\widehat{\mathbb{S}}^n, \mathcal{F})$ and there is an

equality of natural transformations

$$\beta = \eta \circ \widehat{\psi} = (\beta \circ \widehat{\psi}^{-1}|_{\mathbb{R}^n \smallsetminus \overline{U}}) \circ \widehat{\psi}.$$

Therefore, $[\ \beta\ ]$ is in the image of the map in question, $\widehat{\psi}^*[\ \beta \circ \widehat{\psi}^{-1}|_{\mathbb{R}^n \smallsetminus \overline{U}}\ ] = [\ \beta\ ]$, and the map $\widehat{\psi}^*$ is surjective.

Finally, we argue that the map $\widehat{\psi}^*$ is injective. Let $\omega : \mathbb{S}^n \smallsetminus \{0\} \to \mathbb{R}^n$ be a smooth map such that, for some contractible open neighborhood $0 \in V \subset \mathbb{S}^n$, there is an equality of sets $\omega(V \smallsetminus \{0\}) = U$.

Let $\alpha, \alpha' \in \mathrm{Nat}_P(\widehat{\mathbb{S}}^n, \mathcal{F})$ such that precomposing these natural transformations by the $\widehat{\psi}$ yields $\sim_{n'}$-equivalent natural transformations $\alpha \circ \widehat{\psi} \sim_{n'} \alpha' \circ \widehat{\psi}$. Let $H \in \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n, \mathcal{F})$ be the pointed natural transformation witnessing this equivalence.

Then $H \circ (\mathbb{1}_{\widehat{\mathbb{R}}}, \widehat{\omega})$ is a natural transformation between sheaves $\widehat{\mathbb{R}} \times \widehat{\mathbb{S}^n \smallsetminus \{0\}}$ and $\mathcal{F}$ such that H is pointed:

$$H \circ (\mathbb{1}_{\mathbb{R}}, \omega) \circ \widehat{incl}_{\mathbb{R} \times V \smallsetminus \{0\}} = H \circ \widehat{incl_U} \circ (\mathbb{1}_{\mathbb{R}}, \omega) \circ \widehat{incl}_{\mathbb{R} \times V \smallsetminus \{0\}} = P.$$

Thus, the natural transformation $H \circ (\mathbb{1}_{\widehat{\mathbb{R}}}, \widehat{\omega})$ can be extended to all of $\widehat{\mathbb{R}} \times \widehat{\mathbb{S}}^n$ such that the extension $\tilde{H} \in \mathrm{Nat}(\widehat{\mathbb{R}} \times \widehat{\mathbb{S}}^n, \mathcal{F})$ is pointed; $\tilde{H} \circ (\mathbb{1}_{\widehat{\mathbb{R}}}, \widehat{0}) = P$. Thus, $\tilde{H} \in \mathrm{Nat}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{S}}^n, \mathcal{F})$.

Since the natural transformation $\tilde{H}$ extended the homotopy $H \circ (\mathbb{1}_{\mathbb{R}}, \omega)$, it also gave extensions of natural transformation $\alpha \circ \widehat{\psi} \circ \omega$ and $\alpha' \circ \widehat{\psi} \circ \omega$ to natural transformations defined on $\widehat{\mathbb{S}}^n$. The natural transformation $\tilde{H}$ immediately yields an equivalence, $\alpha \circ \widehat{\psi} \circ \omega \sim_n \alpha' \circ \widehat{\psi} \circ \omega$.

Since the composition of smooth maps $\psi \circ \omega : \mathbb{S}^n \smallsetminus \{0\} \to \mathbb{S}^n$ sends the set $V \smallsetminus \{0\}$ to 0, the map can be extended so that $\psi \circ \omega(0) = 0$. Therefore, since V is contractible,

the map $\psi \circ \omega$ is homotopic to the identity $\mathbb{1}_{\mathbb{S}^n}$. Thus,

$$\alpha \sim_n \alpha \circ \widehat{\psi} \circ \omega \sim_n \alpha' \circ \widehat{\psi} \circ \omega \sim_n \alpha'$$

and $\widehat{\psi}^*$ is an injective map.

$\square$

Homotopy groups of a pointed sheaf as a group

Now that is has been shown that our two definitions of the nth homotopy group of a pointed sheaf agree, we will primarily use Definition 3.0.37 to establish the group structure.

Concatenation in $\pi_n(\mathcal{F}, P)$

We will begin by defining concatenation in the collection of pointed natural transformations $\widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$ before showing that concatenation will descend to a binary operation on the set $\pi_n(\mathcal{F}, P)$.

Let $\alpha, \beta \in \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$ be pointed natural transformations. Each has an open set, W_α and W_β respectively, on which the natural transformation is constant (see Observation 3.0.39) and these neighborhoods contain of the set $\overline{\mathbb{R}^n \smallsetminus I \times \mathbb{R}^{n-1}}$.

Now, the compact set $\{0\} \times I^{n-1}$ is contained in the open set W_α. By the tube lemma, there exists a positive value $\epsilon_\alpha^0 > 0$ such that there is set containment,

$$(-\epsilon_\alpha^0, \epsilon_\alpha^0) \times I^{n-1} \subset W_\alpha.$$

In fact, since the complement of the open set W_α is contained within the closed box I^n, the set $(-\epsilon_\alpha^0, \epsilon_\alpha^0) \times \mathbb{R}^{n-1}$ is contained within W_α. Repeating this argument, we find

positive values $\epsilon_\alpha^0, \epsilon_\alpha^1, \epsilon_\beta^0, \epsilon_\beta^1 > 0$ such that there is the set containment

$$((-\epsilon_\alpha^0, \epsilon_\alpha^0) \cup (1 - \epsilon_\alpha^1, 1 + \epsilon_\alpha^1)) \times \mathbb{R}^{n-1} \subset W_\alpha,$$

and

$$((-\epsilon_\beta^0, \epsilon_\beta^0) \cup (1 - \epsilon_\beta^1, 1 + \epsilon_\beta^1)) \times \mathbb{R}^{n-1} \subset W_\beta.$$

Take the positive value $\epsilon := \min(\epsilon_\alpha^0, \epsilon_\alpha^1, \epsilon_\beta^0, \epsilon_\beta^1)$.

Form a two-term open cover of $\mathbb{R}^n$,

$$U = \left(-\infty, \frac{1+\epsilon}{2}\right) \times \mathbb{R}^{n-1} \text{ and } V = \left(\frac{1-\epsilon}{2}, \infty\right) \times \mathbb{R}^{n-1}.$$

Then, there are natural transformations $\alpha \circ (\widehat{2t_1}, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) \in \mathrm{Nat}_P(\widehat{U}, \mathcal{F})$ and $\beta \circ (\widehat{2t_1 - 1}, \mathbb{1}_{r\mathbb{R}^{n-1}}) \in \mathrm{Nat}_P(\widehat{V}, \mathcal{F})$.

By design, we have equality of natural transformations

$$\alpha \circ (\widehat{2t_1}, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) = P \text{ on } \left(\frac{1-\epsilon_\alpha}{2}, \frac{1+\epsilon_\alpha}{2}\right) \times \mathbb{R}^{n-1}$$

and

$$\beta \circ (\widehat{2t_1 - 1}, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) = P \text{ on } \left(\frac{-\epsilon_\beta}{2}, \frac{\epsilon_\beta}{2}\right) \times \mathbb{R}^{n-1}.$$

Since $\left(\frac{1-\epsilon}{2}, \frac{1+\epsilon}{2}\right) \subset \left(\frac{1-\epsilon_\tau}{2}, \frac{1+\epsilon_\tau}{2}\right)$ for $\tau = \alpha$ and β, there is the equality of natural transformations

$$\alpha \circ (\widehat{2t_1}, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) \circ \widehat{incl_{U\cap V}} = P = \beta \circ (\widehat{2t_1 - 1}, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) \circ \widehat{incl_{U\cap V}}$$

and the following diagram commutes

$$\begin{tikzcd}
* \arrow[dr, dotted, "\alpha*\beta", "\exists!"'] \arrow[drr, bend left, "\alpha\circ(\widehat{2t_1},\mathbb{1}_{\widehat{\mathbb{R}}^{n-1}})"] \arrow[ddr, bend right, "\beta\circ(\widehat{2t_1-1},\mathbb{1}_{\widehat{\mathbb{R}}^{n-1}})"'] & & \\
& \mathrm{Nat}_P(\widehat{\mathbb{R}}^n,\mathcal{F}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Nat}_P(\widehat{U},\mathcal{F}) \arrow[d] \\
& \mathrm{Nat}_P(\widehat{V},\mathcal{F}) \arrow[r] & \mathrm{Nat}_P(\widehat{U\cap V},\mathcal{F}).
\end{tikzcd}$$

Since the presheaf $\mathrm{Nat}_P(\widehat{-},\mathcal{F})$ satisfies the descent condition for two-term open covers (Lemma 3.0.30), there exists a unique morphism $\alpha * \beta \in \mathrm{Nat}_P(\widehat{\mathbb{R}}^n,\mathcal{F})$ filling the diagram.

All that remains to be seen is that there is agreement of natural transformations $\alpha * \beta$ and P on some neighborhood of the set $\overline{\mathbb{R}^n \setminus I \times \mathbb{R}^{n-1}}$.

Let $\widetilde{W}_\alpha \subset \mathbb{R}^n$ be the image of the open set W_α under the diffeomorphism $\left(\frac{1}{2}\mathbb{1}_{\mathbb{R}}, \mathbb{1}_{\mathbb{R}^{n-1}}\right)$. Likewise, let $\widetilde{W}_\beta \subset \mathbb{R}^n$ be the image of the open set W_β under the diffeomorphism $((1/2)\mathbb{1}_{\mathbb{R}} - 1/2, \mathbb{1}_{\mathbb{R}^{n-1}})$.

It is immediate that there is an equality of natural transformations $\alpha \circ (2\mathbb{1}_{\widehat{\mathbb{R}}}, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) = P$ on the open set $\widetilde{W}_\alpha$ and that there is an equality of natural transformations $\beta \circ (2\mathbb{1}_{\widehat{\mathbb{R}}} - 1, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) = P$ on the open set $\widetilde{W}_\beta$.

As the natural transformations α and β are pointed, we have equality of natural transformations

$$\alpha \circ (2\mathbb{1}_{\widehat{\mathbb{R}}}, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) \circ \widehat{incl}_{\widetilde{W}_\alpha \cap \widetilde{W}_\beta} = \alpha \circ (2\mathbb{1}_{\widehat{\mathbb{R}}}, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) \circ \widehat{incl}_{\widetilde{W}_\alpha} \circ \widehat{incl}_{\widetilde{W}_\alpha \cap \widetilde{W}_\beta} = P$$

and

$$\beta \circ (2\mathbb{1}_{\widehat{\mathbb{R}}} - 1, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) \circ \widehat{incl}_{\widetilde{W}_\alpha \cap \widetilde{W}_\beta} = P.$$

Thus, the concatenation $\alpha * \beta$ agrees with the natural transformation P on the open set $\widetilde{W}_\alpha \cap \widetilde{W}_\beta$. Also, $\overline{\mathbb{R}^n \setminus I \times \mathbb{R}^{n-1}} \subset \widetilde{W}_\alpha \cap \widetilde{W}_\beta$ since it is a subset of both parts of the intersection. Thus, the concatenation of α and β is constant near the boundary of the

box I^n, that is, $\alpha * \beta \in \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$. Take, $\alpha * \beta$ to be the concatenation of pointed natural transformations $\alpha, \beta \in \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$.

Now that concatenation has been defined, we will show that it determines an operation on the set $\pi_n(\mathcal{F}, P)$.

Lemma 3.0.44. *For equivalence classes* $[\ \alpha\ ], [\ \beta\ ] \in \pi_n(\mathcal{F}, P)$*, the operation defined by*

$$[\ \alpha\ ] * [\ \beta\ ] := [\ \alpha * \beta\ ]$$

is a well-defined binary operation.

Proof. Begin by taking equivalent pointed natural transformations from the equivalence classes $[\ \alpha\ ]$ and $[\ \beta\ ]$ in $\pi_n(\mathcal{F}, P)$. Let $\alpha \sim_{n'} \alpha'$ and $\beta \sim_{n'} \beta'$ in $\widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$ witnessed by homotopies $H_\alpha, H_\beta \in \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n, \mathcal{F})$ respectively. Let $W_\alpha, W_\beta \subset \mathbb{R} \times \mathbb{R}^n$ be open neighborhoods of $\mathbb{R} \times \mathbb{R}^n \setminus I^n$ associated with H_α and H_β respectively.

We will construct a natural transformation $H_\alpha * H_\beta$ which will be a homotopy witnessing the equivalence between natural transformations $\alpha * \beta$ and $\alpha' * \beta *$.

By the tube lemma, there exists positive values $\epsilon_\alpha, \epsilon_\beta > 0$ such that

$$\mathbb{R} \times (1 - \epsilon_\alpha, 1 + \epsilon_\alpha) \times \mathbb{R}^{n-1} \subset W_\alpha$$

and

$$\mathbb{R} \times (-\epsilon_\beta, \epsilon_\beta) \times \mathbb{R}^{n-1} \subset W_\beta.$$

Take the positive value $\epsilon = \min(\epsilon_\alpha, \epsilon_\beta)$.

Take the two-term open cover of $\mathbb{R} \times \mathbb{R}^n$ given by $\{\mathbb{R} \times U, \mathbb{R} \times V\}$ where $U = \left(-\infty, \frac{1+\epsilon}{2}\right) \times \mathbb{R}^{n-1}$ and $V = \left(\frac{1-\epsilon}{2}, \infty\right) \times \mathbb{R}^{n-1}$.

Then, the natural transformations

$$H_\alpha \circ (\mathbb{1}_{\widehat{\mathbb{R}}}, 2\mathbb{1}_{\widehat{\mathbb{R}}}, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) \in \mathrm{Nat}_P(\widehat{\mathbb{R}} \times \widehat{U}, \mathcal{F})$$

and

$$H_\beta \circ (\mathbb{1}_{\widehat{\mathbb{R}}}, 2\mathbb{1}_{\widehat{\mathbb{R}}} - 1, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) \in \mathrm{Nat}_P(\widehat{\mathbb{R}} \times \widehat{V}, \mathcal{F}),$$

on the open set $\mathbb{R} \times (U \cap V)$, there is an equality of natural transformations,

$$H_\alpha \circ (\mathbb{1}_{\widehat{\mathbb{R}}}, 2\mathbb{1}_{\widehat{\mathbb{R}}}, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) = P = H_\beta \circ (\mathbb{1}_{\widehat{\mathbb{R}}}, 2\mathbb{1}_{\widehat{\mathbb{R}}} - 1, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}).$$

So, the following diagram commutes and defines a natural transformation $H_\alpha * H_\beta$ since the presheaf $\mathrm{Nat}_P(\widehat{-}, \mathcal{F})$ satisfies the descent condition for two-term open covers (Lemma 3.0.30):

$$\begin{tikzcd}
* \arrow[dr, dotted, "H_\alpha * H_\beta", "\exists!"'] \arrow[drr, bend left, "H_\alpha \circ (\mathbb{1}_{\widehat{\mathbb{R}}}, 2\mathbb{1}_{\widehat{\mathbb{R}}}, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}})"] \arrow[ddr, bend right, "H_\beta \circ (\mathbb{1}_{\widehat{\mathbb{R}}}, 2\mathbb{1}_{\widehat{\mathbb{R}}} - 1, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}})"'] & & \\
& \mathrm{Nat}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n, \mathcal{F}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Nat}_P(\widehat{\mathbb{R}} \times \widehat{U}, \mathcal{F}) \arrow[d] \\
& \mathrm{Nat}_P(\widehat{\mathbb{R}} \times \widehat{V}, \mathcal{F}) \arrow[r] & \mathrm{Nat}_P(\widehat{\mathbb{R}} \times \widehat{U \cap V}, \mathcal{F}).
\end{tikzcd}$$

In fact, the natural transformation $H_\alpha * H_\beta$ is in $\widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n, \mathcal{F})$ since the images of the open set W_α under the diffeomorphism $(\mathbb{1}_{\mathbb{R}}, \frac{1}{2}\mathbb{1}_{\widehat{\mathbb{R}}}, \mathbb{1}_{\mathbb{R}^{n-1}})$ and the open set W_β under $(\mathbb{1}_{\mathbb{R}}, \frac{1}{2} + \frac{1}{2}\mathbb{1}_{\widehat{\mathbb{R}}}, \mathbb{1}_{\mathbb{R}^{n-1}})$ are both open neighborhoods of $\mathbb{R} \times \overline{\mathbb{R}^n \setminus I^n}$ that have non-empty intersection.

If we precompose the maps from $*$ in the above diagram by the natural transformation $(\widehat{0}, \mathbb{1}_{\widehat{\mathbb{R}}^n})$, we get the following commutative diagram:

$$\begin{tikzcd}
\ast \arrow[drrr, bend left, "H_\alpha \circ (\widehat{0}, 2\mathbb{1}_{\widehat{\mathbb{R}}}, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) = \alpha \circ (2\mathbb{1}_{\widehat{\mathbb{R}}}, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}})"] \arrow[dr, dotted, "(\widehat{0}, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}})^*(H_\alpha * H_\beta) = \alpha * \beta", "\exists !"'] \arrow[ddr, bend right, "H_\beta \circ (\widehat{0}, 2\mathbb{1}_{\widehat{\mathbb{R}}} - 1, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) = \beta \circ (2\mathbb{1}_{\widehat{\mathbb{R}}} - 1, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}})"'] & & & \\
& \mathrm{Nat}_P(\widehat{\mathbb{R}}, \mathcal{F}) \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner"] & & \mathrm{Nat}_P(\widehat{U}, \mathcal{F}) \arrow[d] \\
& \mathrm{Nat}_P(\widehat{V}, \mathcal{F}) \arrow[rr] & & \mathrm{Nat}_P(\widehat{U \cap V}, \mathcal{F}).
\end{tikzcd}$$

By uniqueness, since this diagram is identical to the diagram used to define concatenation, this implies an equality of natural transformations,

$$(H_\alpha * H_\beta) \circ (\widehat{0}, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) = \alpha * \beta.$$

Likewise, there is an equality of natural transformations $(H_\alpha * H_\beta) \circ (\widehat{1}, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) = \alpha' * \beta'$ when the natural transformation $\widehat{0}$ is replaced by $\widehat{1}$. Thus, the natural transformation $H_\alpha * H_\beta$ is a homotopy that witnesses the equivalence $\alpha * \beta \sim_{n'} \alpha' * \beta'$. $\square$

Group structure on $\pi_n(\mathcal{F}, P)$

We will show that the nth homotopy group $\pi_n(\mathcal{F}, P)$ is a group under the operation of concatenation $*$. First, a notion of concatenating smooth maps and two useful lemmas will be established.

Before proceeding, we will define an idea of concatenation on a particular subset of the set of smooth maps $\mathsf{Map}^{\mathsf{sm}}(\mathbb{R}^n, \mathbb{R}^n)$.

Recall from Lemma 3.0.35 the smooth self-map $\mathfrak{g}$ of $\mathbb{R}$ which restricts to the constant maps $\mathfrak{g}|_{(-\infty, \frac{1}{4}]} \equiv 0$ and $\mathfrak{g}|_{[\frac{3}{4}, \infty)} \equiv 1$ and which is strictly increasing on the interval $(1/4, 3/4)$.

The map $\mathfrak{g}$ is smoothly homotopic rel the points $0, 1 \in \mathbb{R}$ via a straight-line homotopy. Moreover, the straight-line homotopy maps any point in the subset $\mathbb{R} \setminus I$ into the subset $\mathbb{R} \setminus I$ for all times.

Now, define the smooth map $\mathfrak{g}^n : \mathbb{R}^n \to \mathbb{R}^n$ by the assignment $\mathfrak{g}^n(t_1, \ldots, t_n) \coloneqq (\mathfrak{g}(t_1), \ldots, \mathfrak{g}(t_n))$. Since the map $\mathfrak{g}$ is smoothly homotopic to the identity map rel the boundary of I, the map $\mathfrak{g}^n$ is also smoothly homotopic to the identity map rel the boundary of I^n via the product of straight-line homotopies. Furthermore, the homotopy maps the subset $\mathbb{R}^n \smallsetminus I^n$ into itself for all times.

We will use this map to define concatenation of certain smooth self-maps of $\mathbb{R}^n$.

Definition 3.0.45. Let $f, f' \in \mathsf{Map}^{\mathsf{sm}}(\mathbb{R}^n, \mathbb{R}^n)$ be smooth maps such that

- $f(\overline{\mathbb{R}^n \smallsetminus I^n}),\ f'(\overline{\mathbb{R}^n \smallsetminus I^n}) \subset \overline{\mathbb{R}^n \smallsetminus I^n}$, and
- $f|_{\{1\}\times\mathbb{R}^{n-1}} = (p, \mathbb{1}_{\mathbb{R}^{n-1}}) = f'|_{\{0\}\times\mathbb{R}^{n-1}}$ for some $p \in \mathbb{R}$.

Let $U = (-\infty, 9/16) \times \mathbb{R}^{n-1}$ and $V = (7/16, \infty) \times \mathbb{R}^{n-1}$. Define the *concatenation of the smooth maps* f and f', denoted by $f * f' \in \mathsf{Map}^{\mathsf{sm}}(\mathbb{R}^n, \mathbb{R}^n)$, to be the unique filler of the following diagram:

$$\begin{tikzcd}
* \arrow[dr, dotted, "f*f'", "\exists!"'] \arrow[drr, bend left, "f\circ\mathfrak{g}^n(2t,\mathbb{1}_{\mathbb{R}^{n-1}})"] \arrow[ddr, bend right, "f'\circ\mathfrak{g}^n(2t-1,\mathbb{1}_{\mathbb{R}^{n-1}})"'] & & \\
& \mathsf{Map}^{\mathsf{sm}}(\mathbb{R}^n, \mathbb{R}^n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Map}^{\mathsf{sm}}(U, \mathbb{R}^n) \arrow[d] \\
& \mathsf{Map}^{\mathsf{sm}}(V, \mathbb{R}^n) \arrow[r] & \mathsf{Map}^{\mathsf{sm}}(U \cap V, \mathbb{R}^n)
\end{tikzcd}$$

Observation 3.0.46. For such maps, $f * f'$ also maps $\overline{\mathbb{R}^n \smallsetminus I^n}$ into itself.

Proposition 3.0.47. *Let $(\mathcal{F}, P)$ be a pointed sheaf on **Man** and let $n \geq 0$. Let $\alpha \in \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$ be a pointed natural transformation. Let $f, f' \in \mathsf{Map}^{\mathsf{sm}}(\mathbb{R}^n, \mathbb{R}^n)$ be smooth, homotopic maps with homotopy $F \in \mathsf{Map}^{\mathsf{sm}}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ such that*

$$f(\mathbb{R}^n \smallsetminus I^n), f'(\mathbb{R}^n \smallsetminus I^n), F(\mathbb{R} \times (\mathbb{R}^n \smallsetminus I^n)) \subset \mathbb{R}^n \smallsetminus I^n.$$

Then, the pointed natural transformations $\alpha \circ f \sim_n' \alpha \circ f' \in \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$ *are homotopic via the homotopy* $\alpha \circ F \in \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n, \mathcal{F})$.

Proof. First, we note several equalities of natural transformations:

$$\alpha \circ f \circ incl_{\mathbb{R}^n \smallsetminus I^n} = \alpha \circ incl_{\mathbb{R}^n \smallsetminus I^n} \circ f \circ incl_{\mathbb{R}^n \smallsetminus I^n} = P;$$

$$\alpha \circ f' \circ incl_{\mathbb{R}^n \smallsetminus I^n} = \alpha \circ incl_{\mathbb{R}^n \smallsetminus I^n} \circ f' \circ incl_{\mathbb{R}^n \smallsetminus I^n} = P;$$

$$\alpha \circ F \circ incl_{\mathbb{R} \times \mathbb{R}^n \smallsetminus I^n} = \alpha \circ incl_{\mathbb{R}^n \smallsetminus I^n} \circ F \circ incl_{\mathbb{R} \times \mathbb{R}^n \smallsetminus I^n} = P;$$

$$\alpha \circ F \circ (\widehat{0}, \mathbb{1}_{\widehat{\mathbb{R}}^n}) = \alpha \circ f \text{ and } \alpha \circ F \circ (\widehat{1}, \mathbb{1}_{\widehat{\mathbb{R}}^n}) = \alpha \circ f'.$$

Now, we argue that the natural transformations $\alpha \circ f$, $\alpha \circ f'$, and $\alpha \circ F$ are constant on a neighborhood of the complement of the n-dimensional box (or $\mathbb{R}$ times the n-dimensional box in the case of $\alpha \circ F$).

Let W_α be the open subset of $\mathbb{R}^n$ associated to the pointed natural transformation α such that $\mathbb{R}^n \smallsetminus I^n \subset W_\alpha$. By assumption, each of the smooth maps f and f' map the subset $\mathbb{R}^n \smallsetminus I^n$ into itself. Thus, the preimage of the open subset $\mathbb{R}^n \smallsetminus I^n$ with respect to f and f' is an open subset of $\mathbb{R}^n$ that contains $\mathbb{R}^n \smallsetminus I^n$ and the compositions $\alpha \circ f$ and $\alpha \circ f'$ are constant on this open subset. Thus, each of $\alpha \circ f$ and $\alpha \circ f'$ is an element of $\widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$. A similar argument shows that $\alpha \circ F$ is an element of the set $\widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n, \mathcal{F})$. $\square$

Proposition 3.0.48. *For pointed natural transformation* $\alpha \in \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$ *and smooth maps* $f, f' \in \mathsf{Map}^{\mathsf{sm}}(\mathbb{R}^n, \mathbb{R}^n)$ *satisfying the conditions of Definition 3.0.45, then there is a homotopy of pointed natural transformations:*

$$\alpha \circ (f * f') \sim_{n'} (\alpha \circ f) * (\alpha \circ f').$$

Proof. Let $W_\alpha \subset \mathbb{R}^n$ be the open neighborhood associated to the natural transformation α such that α is constant on this neighborhood: $\alpha \circ \widehat{incl}_{W_\alpha} = P$.

Since the image of the smooth map $\mathfrak{g}^n(\mathbb{R}^n \smallsetminus [1/4, 3/4]^n) = \partial I^n$ and $f(\partial I^n) \subset \overline{\mathbb{R}^n \smallsetminus I^n} \subset W_\alpha$, the composition $\alpha \circ f \circ \mathfrak{g}^n$ is an element of $\widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$. Likewise, $\alpha \circ f' \circ \mathfrak{g}^n \in \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$.

So, the concatenation $(\alpha \circ f \circ \mathfrak{g}^n) * (\alpha \circ f' \circ \mathfrak{g}^n)$ is defined and, by uniqueness, we have an equality of natural transformations $\alpha \circ (f * f') = (\alpha \circ f \circ \mathfrak{g}^n) * (\alpha \circ f' \circ \mathfrak{g}^n)$ indicated by the following diagram:

$$\begin{tikzcd}
* \arrow[drrr, bend left, "f\circ\mathfrak{g}^n(2t,\mathbb{1}_{\mathbb{R}^{n-1}})"] \arrow[dddr, bend right, "f'\circ\mathfrak{g}^n(2t-1,\mathbb{1}_{\mathbb{R}^{n-1}})"'] \arrow[dr, dotted, "f*f'"'] \arrow[ddrr, dotted, "(\alpha\circ f\circ\mathfrak{g}^n)*(\alpha\circ f'\circ\mathfrak{g}^n)", "\exists!"'] & & & & \\
& \mathrm{Nat}(\widehat{\mathbb{R}}^n, \widehat{\mathbb{R}}^n) \arrow[rr] \arrow[dr, "\alpha"'] \arrow[dd] & & \mathrm{Nat}(\widehat{U}, \widehat{\mathbb{R}}^n) \arrow[dr, "\alpha"] \arrow[dd] & \\
& & \mathrm{Nat}_P(\widehat{\mathbb{R}}^n, \mathcal{F}) \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner"] & & \mathrm{Nat}_P(\widehat{U}, \mathcal{F}) \arrow[dd] \\
& \mathrm{Nat}(\widehat{V}, \widehat{\mathbb{R}}^n) \arrow[rr] \arrow[dr, "\alpha"] & & \mathrm{Nat}(\widehat{U \cap V}, \widehat{\mathbb{R}}^n) \arrow[dr, "\alpha"] & \\
& & \mathrm{Nat}_P(\widehat{V}, \mathcal{F}) \arrow[rr] & & \mathrm{Nat}_P(\widehat{U \cap V}, \mathcal{F}).
\end{tikzcd}$$

Now, since f is a smooth map that maps the subset $\mathbb{R}^n \smallsetminus I^n$ into itself, the composition $\alpha \circ f$ is an element of the set $\widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$, as is the natural transformation $\alpha \circ f \circ \mathfrak{g}^n$. Since the smooth map $\mathfrak{g}^n$ is homotopic to the identity, and this homotopy can be taken such that the subset $\mathbb{R}^n \smallsetminus I^n$ maps into itself for all times, there is a homotopy between the natural transformations $\alpha \circ f$ and $\alpha \circ f \circ \mathfrak{g}^n$ in the set $\widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R} \times \mathbb{R}^n}, \mathcal{F})$. This follows similarly with respect to f'.

By Lemma 3.0.44, concatenating the natural transformations $\alpha \circ f$ and $\alpha \circ f'$ yields a natural transformation that is homotopic to the concatenation of $\alpha \circ f \circ \mathfrak{g}^n$

and $\alpha \circ f' \circ \mathfrak{g}^n$. Thus, we have the following desired equivalence:

$$\alpha \circ (f * f') = (\alpha \circ f \circ \mathfrak{g}^n) * (\alpha \circ f' \circ \mathfrak{g}^n) \sim_{n'} (\alpha \circ f) * (\alpha \circ f').$$

□

Definition 3.0.49. Let $\mathcal{G}$ be a sheaf. For a pointed natural transformation $\beta \in \mathrm{Nat}(\widehat{\mathbb{R}}^n, \mathcal{F})$, define the *reverse* of β to be the natural transformation

$$\overline{\beta} \coloneqq \beta \circ (F_{.5}, \mathbb{1}_{\mathbb{R}^{n-1}}) \in \mathrm{Nat}(\widehat{\mathbb{R}}^n, \mathcal{F}).$$

Theorem 3.0.50. *For a pointed sheaf $(\mathcal{F}, P)$, the nth homotopy group $(\pi_n(\mathcal{F}, P), *)$ is a group.*

Proof. We will begin by showing that $[\ P\ ] \in \pi_n(\mathcal{F}, P)$ is the identity element. First consider the smooth maps $\mathbb{1}_{\mathbb{R}^n}, (1, \mathbb{1}_{\mathbb{R}^{n-1}}) \in \mathsf{Map}^{\mathsf{sm}}(\mathbb{R}^n, \mathbb{R}^n)$. Both maps send the closed set $\overline{\mathbb{R}^n \smallsetminus I^n}$ into itself and have the following equality of restrictions,

$$\mathbb{1}_{\mathbb{R}^n}|_{\{1\} \times \mathbb{R}^{n-1}} = (1, \mathbb{1}_{\mathbb{R}^{n-1}})|_{\{0\} \times \mathbb{R}^{n-1}}.$$

So, the concatenation $\mathbb{1}_{\mathbb{R}^n} * (1, \mathbb{1}_{\mathbb{R}^{n-1}}) \in \mathsf{Map}^{\mathsf{sm}}(\mathbb{R}^n, \mathbb{R}^n)$ is defined and maps the set $\overline{\mathbb{R}^n \smallsetminus I^n}$ into itself. Also, due to convexity of $\mathbb{R}$, there is a homotopy between the maps $\mathbb{1}_{\mathbb{R}}$ and $\mathbb{1}_{\mathbb{R}} * 1$ and thus a homotopy $G \in \mathsf{Map}^{\mathsf{sm}}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ between the maps $\mathbb{1}_{\mathbb{R}^n}$ and $\mathbb{1}_{\mathbb{R}^n} * (1, \mathbb{1}_{\mathbb{R}^{n-1}})$ such that $G(\mathbb{R} \times \overline{\mathbb{R}^n \smallsetminus I^n}) \subset \overline{\mathbb{R}^n \smallsetminus I^n}$.

Let $\alpha \in \mathrm{Nat}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$ be a pointed natural transformation. By Proposition 3.0.47, there is a homotopy $\alpha \circ G \in \mathrm{Nat}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n, \mathcal{F})$ witnessing the equivalence

$$\alpha = \alpha \circ \mathbb{1}_{\mathbb{R}^n} \sim_n' \alpha \circ (\mathbb{1}_{\mathbb{R}^n} * (1, \mathbb{1}_{\mathbb{R}^{n-1}})).$$

By Proposition 3.0.48, composition and concatenation commute and there is the following equivalence:

$$\alpha \circ (\mathbb{1}_{\mathbb{R}^n} * (1, \mathbb{1}_{\mathbb{R}^{n-1}})) \sim_n' (\alpha \circ \mathbb{1}_{\mathbb{R}^n}) * (\alpha \circ (1, \mathbb{1}_{\mathbb{R}^{n-1}})) = \alpha * P.$$

So, there is an equality of equivalence classes $[\ \alpha\ ] * [\ P\ ] = [\ \alpha\ ]$. A similar argument shows that $[\ P\ ] * [\ \alpha\ ] = [\ \alpha\ ]$ and the equivalence class $[\ P\ ]$ is an identity element.

We will now go about showing that each equivalence class in $\pi_n(\mathcal{F}, P)$ has an inverse element with respect to the concatenation operation.

Consider the maps $\mathbb{1}_{\mathbb{R}^n}, \overline{\mathbb{1}}_{\mathbb{R}^n} \in \mathsf{Map}^{\mathsf{sm}}(\mathbb{R}^n, \mathbb{R}^n)$. Both map $\mathbb{R}^n \smallsetminus I^n$ into itself and have the following restriction condition:

$$\mathbb{1}_{\mathbb{R}^n}|_{\{1\}\times\mathbb{R}^{n-1}} = \overline{\mathbb{1}}_{\mathbb{R}^n}|_{\{0\}\times\mathbb{R}^{n-1}}.$$

So, the concatenation $\mathbb{1}_{\mathbb{R}^n} * \overline{\mathbb{1}}_{\mathbb{R}^n} \in \mathsf{Map}^{\mathsf{sm}}(\mathbb{R}^n, \mathbb{R}^n)$ is defined and maps $\mathbb{R}^n \smallsetminus I^n$ into itself. Since the maps $\mathbb{1}_{\mathbb{R}} * \overline{\mathbb{1}}_{\mathbb{R}}, 0 \in \mathsf{Map}^{\mathsf{sm}}(\mathbb{R}, \mathbb{R})$ are homotopic, there is a homotopy between the maps $\mathbb{1}_{\mathbb{R}^n} * \overline{\mathbb{1}}_{\mathbb{R}^n}$ and $(0, \mathbb{1}_{\mathbb{R}^{n-1}})$. Such a homotopy $H \in \mathsf{Map}^{\mathsf{sm}}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ can be taken such that $H(\mathbb{R} \times \overline{\mathbb{R}^n \smallsetminus I^n}) \subset \overline{\mathbb{R}^n \smallsetminus I^n}$.

Let $\alpha \in \mathrm{Nat}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$ be an equivalence class. By Proposition 3.0.47, there is an equivalence between pointed natural transformations,

$$\alpha \circ (\mathbb{1}_{\mathbb{R}^n} * \overline{\mathbb{1}}_{\mathbb{R}^n}) \sim_{n'} \alpha \circ (0, \mathbb{1}_{\mathbb{R}^{n-1}}) = P.$$

By Proposition 3.0.48, since concatenation and composition commute, there is an equivalence between pointed natural transformations,

$$\alpha \circ (\mathbb{1}_{\mathbb{R}^n} * \overline{\mathbb{1}}_{\mathbb{R}^n}) \sim_{n'} (\alpha \circ \mathbb{1}_{\mathbb{R}^n}) * (\alpha \circ \overline{\mathbb{1}}_{\mathbb{R}^n}) = \alpha * \overline{\alpha}.$$

So, there are equalities of equivalence classes $[\ \alpha\ ] * [\ \overline{\alpha}\ ] = [\ P\ ]$ and, similarly, $[\ \overline{\alpha}\ ] * [\ \alpha\ ] = [\ P\ ]$. Thus, each element $[\ \alpha\ ]$ of $\pi_n(\mathcal{F}, P)$ has an inverse element: $[\ \alpha\ ]^{-1} = [\ \overline{\alpha}\ ]$.

For associativity, let $\alpha, \beta, \gamma \in \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$ be given pointed natural transformations. We will use the smooth map $\mathfrak{g} \in \mathsf{Map}^{\mathsf{sm}}(\mathbb{R}, \mathbb{R})$ that was used in the proof of Lemma 3.0.35.

The two possible orders of concatenating the three natural transformations yields the following pointed natural transformations in $\widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$:

$$(\alpha * \beta) * \gamma = \begin{cases} \alpha \circ (4t_1, t_2, \ldots, t_n) & , -\infty < t_1 \leq 1/4 \\ \beta \circ (4t_1 - 1, t_2, \ldots, t_n) & , 1/4 \leq t_1 \leq 1/2 \\ \gamma \circ (2t_1 - 1, t_2, \ldots, t_n) & , 1/2 \leq t_1 < \infty \end{cases}$$

and

$$\alpha * (\beta * \gamma) = \begin{cases} \alpha \circ (2t_1, t_2, \ldots, t_n) & , -\infty < t_1 \leq 1/2 \\ \beta \circ (4t_1 - 2, t_2, \ldots, t_n) & , 1/2 \leq t_1 \leq 3/4 \\ \gamma \circ (4t_1 - 3, t_2, \ldots, t_n) & , 3/4 \leq t_1 < \infty. \end{cases}$$

Using properties of $\mathbb{R}$ and the sheaf condition, we will show that these two natural transformations in $\widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}}^n, \mathcal{F})$ are homotopic.

First, let $r \in \mathsf{Map}^{\mathsf{sm}}(\mathbb{R}, \mathbb{R})$ be a smooth map that is monotonically increasing and sends an open neighborhood of $(-\infty, 0]$ to 0, an open neighborhood of $[1, \infty)$ to 1, an open neighborhood of 1/4 to 1/2, and an open neighborhood of 1/2 to 3/4. Such

a smooth map is given by

$$r(t) = \begin{cases} (1/2)\mathfrak{g}(4t) & , -\infty < t \leq 1/4 \\ (1/4)\mathfrak{g}(4t-1) + 1/2 & , 1/4 \leq t \leq 1/2 \\ (1/4)\mathfrak{g}(2t-1) + 3/4 & , 1/2 \leq t < \infty. \end{cases}$$

Since the smooth maps r and $\mathbb{1}_{\mathbb{R}}$ both fix 0 and 1 and $\mathbb{R}$ is convex, r is homotopic to $\mathbb{1}_{\mathbb{R}}$ relative to the points 0 and 1. Thus, the natural transformations $\alpha * (\beta * \gamma) \circ (r, \mathbb{1}_{\mathbb{R}^{n-1}}) \sim_{n'} \alpha * (\beta * \gamma)$ are homotopic.

Now, consider the maps $4t$ and $2r(t)$ from the interval $[0, 1/4]$ to $[0, 1]$. The maps agree on the endpoints of the interval and are thus homotopic. So, there is a smooth homotopy between these maps relative to the endpoints which takes value in $[0, 1]$. Smoothly extend this homotopy, called H_α, such that it has domain $\mathbb{R} \times (-\infty, 1/4]$. Construct H_α such that on $\mathbb{R} \times (-\infty, 0]$, the map takes value outside of the open interval $(0, 1)$.

Likewise, $(4t - 1) \sim \mathfrak{g}(4r(t) - 2))$ via some $H_\beta \in \mathsf{Map}^{\mathsf{sm}}(\mathbb{R} \times [1/4, 1/2], [0, 1])$ relative to the endpoints of the interval $[1/4, 1/2]$. Also, the maps $(2t-1) \sim (4r(t)-3)$ are homotopic via some $H_\gamma \in \mathsf{Map}^{\mathsf{sm}}(\mathbb{R} \times [1/2, \infty), \mathbb{R})$ relative to the points 1/2 and 1 in $[1/2, \infty)$. Construct H_γ such that on $\mathbb{R} \times [1, \infty)$, the map takes value outside of the open interval $(0, 1)$.

Returning to our natural transformation view, the natural transformations $\alpha \circ (\widehat{H}_\alpha, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}})$ and $\beta \circ (\widehat{H}_\beta, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}})$ agree on the open set $\mathbb{R} \times (1/4 - \epsilon, 1/4 + \epsilon) \times \mathbb{R}^{n-1}$ for some $\epsilon > 0$. This is since the natural transformations α and β are constant on some open set containing the boundary of the cube I^n. Since the presheaf $\mathcal{F}$ is a sheaf and Lemma 3.0.30, the natural transformations $\alpha \circ (\widehat{H}_\alpha, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}})$ and $\beta \circ (\widehat{H}_\beta, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}})$ can be glued together.

Similarly, $\beta\circ(\widehat{H}_\beta, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}})$ and $\gamma\circ(\widehat{H}_\gamma, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}})$ agree on a neighborhood of $\mathbb{R}\times\{1/2\}\times\mathbb{R}^{n-1}$. By Lemma 3.0.30, the presheaf $\mathrm{Nat}_P(\widehat{-}, \mathcal{F})$ satisfies the descent condition for two-term open covers and the homotopy $H \in \mathrm{Nat}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}, \mathcal{F})$ between natural transformations $(\alpha * \beta) * \gamma$ and $\alpha * (\beta * \gamma) \circ r$ is constructed

$$H = \begin{cases} \alpha \circ (\widehat{H}_\alpha, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) & , \mathbb{R} \times (-\infty, 1/4] \times \mathbb{R}^{n-1} \\ \beta \circ (\widehat{H}_\beta, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) & , \mathbb{R} \times [1/4, 1/2] \times \mathbb{R}^{n-1} \\ \gamma \circ (\widehat{H}_\gamma, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}}) & , \mathbb{R} \times [1/2, \infty) \times \mathbb{R}^{n-1}. \end{cases}$$

The natural transformation is pointed, i.e., $H \in \widetilde{\mathrm{Nat}}_P(\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n, \mathcal{F})$, since each of the original natural transformations α, β, and γ are pointed and the constructed homotopies of Euclidean space, $(\widehat{H}_\alpha, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}})$, $(\widehat{H}_\beta, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}})$ and $(\widehat{H}_\gamma, \mathbb{1}_{\widehat{\mathbb{R}}^{n-1}})$ map points outside of the box I^n outside of the box.

Therefore, there is an equivalence of natural transformations

$$(\alpha * \beta) * \gamma \sim_{n'} \alpha * (\beta * \gamma) \circ r \sim_{n'} \alpha * (\beta * \gamma)$$

and we arrive at the equality of equivalence classes,

$$([\ \alpha\ ] * [\ \beta\ ]) * [\ \gamma\ ] = [\ \alpha\ ] * ([\ \beta\ ] * [\ \gamma\ ]).$$

□

Horizontal homotopy groups

We now define horizontal homotopy groups of a manifold endowed with a distribution. Per the previous sections of this chapter, we only need to define a

pointed sheaf and then take the sheaf's homotopy groups.

Let (M,Δ) be a manifold endowed with a distribution. Define a presheaf $\mathcal{F}_{(M,\Delta)}$ by

$$\begin{aligned}
\mathcal{F}_{(M,\Delta)} : \mathbf{Man}^{\mathsf{op}} &\to \mathbf{Set} \\
N &\mapsto \mathsf{Map}^{\mathsf{H}}\big(N,(M,\Delta)\big) \\
(N' \xrightarrow{g} N) &\mapsto \left(\begin{array}{rcl} \mathsf{Map}^{\mathsf{H}}\big(N,(M,\Delta)\big) & \xrightarrow{g^*} & \mathsf{Map}^{\mathsf{H}}\big(N',(M,\Delta)\big) \\ f & \mapsto & f\circ g \end{array}\right).
\end{aligned}$$

This presheaf is well-defined as precomposing horizontal maps by smooth maps yields horizontal maps (Lemma 2.0.8). Further, $\mathcal{F}_{(M,\Delta)}$ will send the identity map $\mathbb{1}_N$ to the identity map $\mathbb{1}_{\mathsf{Map}^{\mathsf{H}}(N,(M,\Delta))}$ and compositions to compositions: for smooth maps $g \in \mathsf{Map}^{\mathsf{sm}}(N',N)$ and $h \in \mathsf{Map}^{\mathsf{sm}}(N'',N')$ and horizontal map $f \in \mathsf{Map}^{\mathsf{H}}\big(N,(M,\Delta)\big)$,

$$(g\circ h)^*(f) = (f\circ g)\circ h = h^*(f\circ g) = h^*\circ g^*(f).$$

Lemma 3.0.51. *The presheaf $\mathcal{F}_{(M,\Delta)} : \boldsymbol{Man}^{\mathsf{op}} \longrightarrow \boldsymbol{Set}$ is a sheaf.*

Proof. Let $N \in \mathbf{Man}$ with two-term open cover $\{U,V\}$. Let $Z \in \mathbf{Set}$ be a set with maps such that the following diagram commutes:

$$\begin{tikzcd}
Z \arrow[dr, dotted, "\varphi"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& \mathsf{Map}^{\mathsf{H}}\big(N,(M,\Delta)\big) \arrow[r] \arrow[d] & \mathsf{Map}^{\mathsf{H}}\big(U,(M,\Delta)\big) \arrow[d] \\
& \mathsf{Map}^{\mathsf{H}}\big(V,(M,\Delta)\big) \arrow[r] & \mathsf{Map}^{\mathsf{H}}\big(U\cap M,(M,\Delta)\big).
\end{tikzcd}$$

Then, for every element $z \in Z$, there are horizontal maps $f(z) : U \to (M,\Delta)$ and $g(z) : V \to (M,\Delta)$ that agree on $U \cap V$. We then can define a morphism

$\varphi : Z \to \mathsf{Map}^{\mathsf{sm}}(N, M)$ into the collection of smooth maps from N to M by

$$\varphi(z)(x) := \begin{cases} f(z)(x) & \text{if} \quad x \in U \\ g(z)(x) & \text{if} \quad x \in V. \end{cases}$$

The filling φ is unique since the representable presheaf $\widehat{M}$ is a sheaf.

It remains to be shown that the smooth map $\varphi(z)$ is horizontal. But, since the restrictions $\varphi(z)|_U$ and $\varphi(z)|_V$ are horizontal, and the open subsets U and V cover the manifold N, it is immediate that $\varphi(z)$ is horizontal as well.

So, the presheaf $\mathcal{F}_{(M,\Delta)}$ satisfies the descent condition for two-term open covers. We will now check that it satisfies the descent condition for sequential covers.

For a manifold N, let $\{U_i\}$ be a sequential open cover of N. Suppose there exists a set Z and maps of sets f_i such that the following solid diagram commutes:

$$\begin{tikzcd}
Z \arrow[d, dashed, "\exists!"'] \arrow[rrrr, bend left, "f_i"] \arrow[rrrrrrrr, bend left, "f_1"] \arrow[dd, dotted, bend right, "f"'] & & & & & & & & \\
\mathcal{F}_{(M,\Delta)}(N) \arrow[r] \arrow[d, hook] & \dots \arrow[rrr] & & & \mathcal{F}_{(M,\Delta)}(U_i) \arrow[r] \arrow[d, hook] & \dots \arrow[rrr] & & & \mathcal{F}_{(M,\Delta)}(U_1) \arrow[d, hook] \\
\hat{M}(N) \arrow[r] & \dots \arrow[rrr] & & & \hat{M}(U_i) \arrow[r] & \dots \arrow[rrr] & & & \hat{M}(U_1).
\end{tikzcd}$$

As is indicated in the solid diagram, there are injective maps from each set of horizontal maps on an open subset to the set of smooth maps on the same open subset. Since the representable presheaf $\hat{M}$ is a sheaf, there exists a unique map f from the set Z to $\hat{M}(N)$, indicated by the dotted arrow, such that this diagram commutes.

Now, we will argue that the map of sets f maps into the set of horizontal maps on the set N. For each point $z \in Z$, the map $f(z) : N \to M$ is a smooth map such that, for each index i, the restriction $f(z)|_{U_i}$ agrees with the map $f_i(z)$. By assumption, the smooth map $f_i(z)$ is a horizontal map from U_i into (M, Δ), so the smooth map

$f(z)|_{U_i}$ is horizontal on U_i. Since the open subsets U_i cover the manifold N, the smooth map $f(z)$ is horizontal on N and the map f factors through $\mathcal{F}_{(M,\Delta)}(N)$, as is indicated by the dashed arrow in the diagram.

Finally, note that f is the unique map that fills the diagram as it is the unique map from Z to $\hat{M}(N)$ that allows the diagram to commute. By Proposition 3.0.19, the presheaf $\mathcal{F}_{(M,\Delta)}$ is a sheaf.

□

Now, we should stipulate a natural transformation $\widehat{*} \to \mathcal{F}_{(M,\Delta)}$ to upgrade the sheaf $\mathcal{F}_{(M,\Delta)}$ to a pointed sheaf. Let $p \in M$ be a base point for the manifold M. Then, $p : * \to (M, \Delta)$ is a horizontal map into the manifold with distribution (M, Δ). Thus, via Yoneda lemma, there is a natural transformation $\widehat{p} : \widehat{*} \to \mathcal{F}_{(M,\Delta)}$ associated to the point p. Thus, we will take $(\mathcal{F}_{(M,\Delta)}, \widehat{p})$ to be the pointed sheaf associated to (M, Δ) with base point p.

Definition 3.0.52. For a manifold endowed with a distribution (M, Δ) with a base point p, the *nth horizontal homotopy group* is

$$\pi_n^{\mathsf{H}}((M, \Delta), p) := \pi_n(\mathcal{F}_{(M,\Delta)}, \widehat{p}).$$

This definition agrees with the definition of smooth horizontal homotopy groups provided in Definition 4.1 in [6]. As is stated in the following lemma, the horizontal homotopy groups can be defined in terms of horizontal maps from the n-sphere and homotopies with domain $\mathbb{R} \times \mathbb{S}^n$. The argument for this lemma will closely resemble the proof for Lemma 3.0.36.

Lemma 3.0.53. *For any manifold endowed with a distribution* (M, Δ) *and any point*

$p \in M$, there is a canonical bijection of sets

$$\pi_n^{\mathsf{H}}\big((M,\Delta),p\big) \cong \mathsf{Map}_p^{\mathsf{H}}\big(\mathbb{S}^n,(M,\Delta)\big)/\sim,$$

where two based horizontal maps in $\mathsf{Map}_p^{\mathsf{H}}\big(\mathbb{S}^n,(M,\Delta)\big)$ are equivalent if there exists a horizontal homotopy in $\mathsf{Map}_p^{\mathsf{H}}\big(\mathbb{R}\times\mathbb{S}^n,(M,\Delta)\big)$ between them, that is, $f_0 \sim f_1$ means there is an element $H \in \mathsf{Map}_p^{\mathsf{H}}\big(\mathbb{R}\times\mathbb{S}^n,(M,\Delta)\big)$ for which the restrictions $H|_{\{0\}\times\mathbb{S}^n} = f_0$ and $H|_{\{1\}\times\mathbb{S}^n} = f_1$.

Proof. The set of horizontal maps from $\mathbb{S}^n$ into (M,Δ) based at the point p is denoted $\mathsf{Map}_p^{\mathsf{H}}\big(\mathbb{S}^n,(M,\Delta)\big)$ and is defined using Definition 3.0.27 with respect to the following diagram of constant smooth maps:

$$\begin{tikzcd} * \arrow[r, "p"] \arrow[d, "0"'] & (M,\Delta). \\ \mathbb{S}^n & \end{tikzcd}$$

Via the Yoneda embedding, there are canonical bijections of sets

$$\mathsf{Map}^{\mathsf{H}}\big(\mathbb{S}^n,(M,\Delta)\big) \cong \mathrm{Nat}\big(\widehat{\mathbb{S}}^n,\mathcal{F}_{(M,\Delta)}\big) \text{ and } \mathsf{Map}^{\mathsf{H}}\big(*,(M,\Delta)\big) \cong \mathrm{Nat}\big(\widehat{*},\mathcal{F}_{(M,\Delta)}\big).$$

As the set of pointed horizontal maps $\mathsf{Map}_p^{\mathsf{H}}\big(\mathbb{S}^n,(M,\Delta)\big)$ and the set of pointed natural transformations $\mathrm{Nat}_{\widehat{p}}\big(\widehat{\mathbb{S}}^n,\mathcal{F}_{(M,\Delta)}\big)$ are both defined via pullback of bijective sets, they too are in bijection:

$$\begin{tikzcd}
\mathsf{Map}^{\mathsf{H}}_{p}\big(\mathbb{S}^n,(M,\Delta)\big) \arrow[rr] \arrow[dr, dotted, "j", "\exists\,\cong\ \lrcorner"'] \arrow[dd] & & * \arrow[dr, "\cong"] \arrow[dd, "\{p\}"' near start] & \\
& \mathrm{Nat}_{\widehat{p}}\big(\widehat{\mathbb{S}}^n,\mathcal{F}_{(M,\Delta)}\big) \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner"] & & * \arrow[dd, "\{\widehat{p}\}"] \\
\mathsf{Map}^{\mathsf{H}}\big(\mathbb{S}^n,(M,\Delta)\big) \arrow[rr, "0^*" near end] \arrow[dr, "\cong"] & & \mathsf{Map}^{\mathsf{H}}\big(*,(M,\Delta)\big) \arrow[dr, "\cong"] & \\
& \mathrm{Nat}\big(\widehat{\mathbb{S}}^n,\mathcal{F}_{(M,\Delta)}\big) \arrow[rr, "\widehat{0}^*"] & & \mathrm{Nat}\big(\widehat{*},\mathcal{F}_{(M,\Delta)}\big).
\end{tikzcd} \tag{3.8}$$

By a similar argument, there is a bijection

$$j : \mathsf{Map}^{\mathsf{H}}_{p}\big(\mathbb{R}\times\mathbb{S}^n,(M,\Delta)\big) \cong \mathrm{Nat}_{\widehat{p}}\big(\widehat{\mathbb{R}}\times\widehat{\mathbb{S}}^n,\mathcal{F}_{(M,\Delta)}\big)$$

and the following diagram commutes:

$$\begin{tikzcd}
\mathsf{Map}^{\mathsf{H}}_{p}\big(\mathbb{R}\times\mathbb{S}^n,(M,\Delta)\big) \arrow[r, "j", "\cong"'] \arrow[d, "ev_{0,1}"] & \mathrm{Nat}_{\widehat{p}}\big(\widehat{\mathbb{R}}\times\widehat{\mathbb{S}}^n,\mathcal{F}_{(M,\Delta)}\big) \arrow[d, "ev_{\widehat{0},\widehat{1}}"] \\
\mathsf{Map}^{\mathsf{H}}_{p}\big(\mathbb{S}^n,(M,\Delta)\big)\times\mathsf{Map}^{\mathsf{H}}_{p}\big(\mathbb{S}^n,(M,\Delta)\big) \arrow[r, "j", "\cong"'] & \mathrm{Nat}_{\widehat{p}}\big(\widehat{\mathbb{S}}^n,\mathcal{F}_{(M,\Delta)}\big)\times\mathrm{Nat}_{\widehat{p}}\big(\widehat{\mathbb{S}}^n,\mathcal{F}_{(M,\Delta)}\big).
\end{tikzcd} \tag{3.9}$$

Here, the maps $ev_{0,1}$ and $ev_{\widehat{0},\widehat{1}}$ are evaluation maps:

$$\begin{array}{rccl}
ev_{0,1}: & \mathsf{Map}^{\mathsf{H}}_{p}\big(\mathbb{R}\times\mathbb{S}^n,(M,\Delta)\big) & \to & \mathsf{Map}^{\mathsf{H}}_{p}\big(\mathbb{S}^n,(M,\Delta)\big)\times\mathsf{Map}^{\mathsf{H}}_{p}\big(\mathbb{S}^n,(M,\Delta)\big) \\
& H & \mapsto & \big(H\circ(0,\mathbb{1}_{S^n}),H\circ(1,\mathbb{1}_{S^n})\big).
\end{array}$$

$$\begin{array}{rccl}
ev_{\widehat{0},\widehat{1}}: & \mathrm{Nat}_{\widehat{p}}\big(\widehat{\mathbb{R}}\times\widehat{\mathbb{S}}^n,\mathcal{F}_{(M,\Delta)}\big) & \to & \mathrm{Nat}_{\widehat{p}}\big(\widehat{\mathbb{S}}^n,\mathcal{F}_{(M,\Delta)}\big)\times\mathrm{Nat}_{\widehat{p}}\big(\widehat{\mathbb{S}}^n,\mathcal{F}_{(M,\Delta)}\big) \\
& H & \mapsto & \big(H\circ(\widehat{0},\mathbb{1}_{\widehat{S}^n}),H\circ(\widehat{1},\mathbb{1}_{\widehat{S}^n})\big).
\end{array}$$

We now define a map between the sets $\pi_n^{\mathsf{H}}((M,\Delta),p)$ and $\mathsf{Map}_p^{\mathsf{H}}\big(\mathbb{S}^n,(M,\Delta)\big)/\sim$:

$$\begin{array}{rccc} j: & \mathsf{Map}_p^{\mathsf{H}}\big(\mathbb{S}^n,(M,\Delta)\big)/\sim & \to & \pi_n^{\mathsf{H}}((M,\Delta),p) \\ & [\ \alpha\ ] & \mapsto & [\ j\alpha\ ]. \end{array}$$

We first argue that this map is well-defined.

Two pointed maps $\alpha,\beta \in \mathsf{Map}_p^{\mathsf{H}}\big(\mathbb{S}^n,(M,\Delta)\big)$ are equivalent in the domain of the map j if there exists a horizontal homotopy $H \in \mathsf{Map}_p^{\mathsf{H}}\big(\mathbb{R}\times\mathbb{S}^n,(M,\Delta)\big)$ such that $ev_{0,1}(H) = (\alpha,\beta)$. Since diagram (3.9) commutes, the image of the homotopy H under the bijection j is a pointed natural transformation $jH \in \mathrm{Nat}_{\widehat{p}}(\widehat{\mathbb{R}}\times\widehat{\mathbb{S}}^n,\mathcal{F}_{(M,\Delta)})$. Thus, we have an equality of pairs of pointed natural transformations:

$$ev_{\widehat{0},\widehat{1}}(\widehat{H}) = (\widehat{H}\circ(\widehat{0},\mathbb{1}_{\widehat{S}^n}),\widehat{H}\circ(\widehat{1},\mathbb{1}_{\widehat{S}^n})) = (j\alpha,j\beta).$$

So, the pointed natural transformations $j\alpha$ and $j\beta$ are homotopic via the pointed natural transformation jH, and the map j is well-defined.

We now define a map back between the sets,

$$\begin{array}{rccc} j^{-1}: & \pi_n^{\mathsf{H}}((M,\Delta),p) & \to & \mathsf{Map}_p^{\mathsf{H}}\big(\mathbb{S}^n,(M,\Delta)\big)/\sim \\ & [\ \widehat{\alpha}\ ] & \mapsto & [\ j^{-1}(\widehat{\alpha})\ ]. \end{array}$$

As was the case with the map j, using diagram (3.9) yields that the map j^{-1} is well-defined. These maps are indeed inverses of each other as they are defined in terms of the isomorphism guaranteed by diagram (3.8). Thus, there is a bijection between $\pi_n^{\mathsf{H}}((M,\Delta),p)$ and $\mathsf{Map}_p^{\mathsf{H}}\big(\mathbb{S}^n,(M,\Delta)\big)/\sim$ and the sheaf definition of horizontal homotopy groups, that is Definition 3.0.33 applied to the sheaf $\mathcal{F}_{(M,\Delta)}$, agrees the standard definition. □

Horizontal homotopy groups will be our primary tool for reporting the probing of contact 3-manifolds by horizontal maps.

Observation 3.0.54. For any manifold endowed with a distribution (M, Δ) with a base point $p \in M$, there is an obvious forgetful homomorphism,

$$\pi_n^{\mathsf{H}}((M,\Delta),p) \longrightarrow \pi_n(M,p),$$

since every horizontal map is also smooth.

Observation 3.0.55. If (M, Δ) is a manifold endowed with an integrable distribution, the horizontal homotopy groups at a chosen base point $p \in M$ agree with the homotopy groups of the leaf L that the point lies in:

$$\pi_n^{\mathsf{H}}((M,\Delta),p) \cong \pi_n(L,p).$$

This follows from Proposition 2.0.33 as any horizontal map will have image that remains in the same leaf as the base point.

The base point will often be suppressed when it is not of utmost importance. In fact for any contact manifold, the horizontal homotopy groups are the same no matter the choice of base point, as expressed in the next lemma.

In the following proof, we will use a definition of concatenation of maps that are doubly-pointed, not just pointed. Via the isomorphisms in Lemma 3.0.36 and Lemma 3.0.43 and the definition of concatenation on $\pi_n^{\mathsf{H}}((M,\xi),p)$ (Lemma 3.0.44), concatenation is defined for pointed horizontal maps. Let

$$F : (\mathbb{S}^n, (-1,0,\ldots,0), (1,0,\ldots,0)) \to (M, p, p')$$

and

$$G : (\mathbb{S}^n, (-1,0,\ldots,0), (1,0,\ldots,0)) \to (M, p', p'')$$

be doubly-pointed, horizontal maps. The concatenation $F * G$ is the horizontal map resulting from the concatenation process applied to the maps F and G where p' is taken to be the base point for both maps. The horizontal map $F * G$ is based at p. The map F should be thought of as a means of translating based maps at p' to based maps at p.

Lemma 3.0.56. *For a manifold endowed with a bracket-generating distribution (M,ξ) and two points $p, p' \in M$, a choice of horizontal path from p to p' determines an isomorphism between the nth horizontal homotopy group based at p and the nth horizontal homotopy group based at p':*

$$\pi_n^{\mathsf{H}}((M,\xi),p) \cong \pi_n^{\mathsf{H}}((M,\xi),p').$$

Proof. Chow-Rashevskii theorem implies that the manifold endowed with a bracket-generating distribution (M,ξ) is horizontally path connected (1.2B [15]). Thus, there exists a horizontal path $\gamma : I \to (M,\xi)$ that connects the point $\gamma(0) = p$ to the point $\gamma(1) = p'$. Further, we can assume that all derivatives of the path γ vanish at the points 0 and 1. We will use this horizontal path to construct an isomorphism. For this proof, we will treat $(-1,0,\ldots,0) \in \mathbb{S}^n$ as the base point for the n-sphere.

First, we will note that the horizontal path γ can be used to define a pair of horizontal maps from the n-sphere. Let $\mathsf{pr} : \mathbb{S}^n \to [-1,1]$ be the projection map of the n-sphere onto its first coordinate. Now, the composition of maps $\gamma_n := \gamma \circ \left(\frac{1}{2}\mathsf{pr} + \frac{1}{2}\right) : \mathbb{S}^n \to (M,\xi)$ is a horizontal map. The map γ_n sends the point $(-1,0,\ldots,0)$ to p and the point $(1,0,\ldots,0)$ to p'.

Now, for the horizontal path γ, the reverse $\overline{\gamma}$ of the path γ be given by $\overline{\gamma}(t) = \gamma(1-t)$. The reverse of the map γ_n is $\overline{\gamma_n} := \overline{\gamma} \circ \left(\frac{1}{2}\mathsf{pr} + \frac{1}{2}\right)$. The map $\overline{\gamma_n}$ sends the point $(-1,0,\ldots,0)$ to p' and the point $(1,0,\ldots,0)$ to p.

Since the concatenation of horizontal paths $\gamma * \overline{\gamma}$ is horizontally null-homotopic, the horizontal map $\gamma_n * \overline{\gamma_n}$ based at the point p is horizontally null-homotopic. Similarly, the horizontal map $\overline{\gamma_n} * \gamma_n$ based at p' is also horizontally null-homotopic.

Let the map $\alpha : \mathbb{S}^n \to (M, \xi)$ represent an element of $\pi_n^{\mathsf{H}}((M, \xi), p)$. The concatenation $\overline{\gamma_n} * \alpha$ is a representative of an element in $\pi_n^{\mathsf{H}}((M, \xi), p')$. So, we define the map between horizontal homotopy groups by

$$\begin{aligned} \Gamma : \quad \pi_n^{\mathsf{H}}((M, \xi), p) &\to \pi_n^{\mathsf{H}}((M, \xi), p') \\ [\alpha] \qquad &\mapsto [\overline{\gamma_n} * \alpha] \end{aligned}$$

The map Γ is well-defined since concatenation of elements in $\pi_n^{\mathsf{H}}((M, \xi), p')$ is well-defined (Lemma 3.0.44).

We will now argue that the map Γ is bijective. First, we will show that the map Γ is injective. Take horizontal maps α and α' representing homotopy classes in $\pi_n^{\mathsf{H}}((M, \xi), p)$ and assume that the concatenations with the horizontal map $\overline{\gamma_n}$ which are based at p' are horizontally homotopic:

$$\overline{\gamma_n} * \alpha \simeq \overline{\gamma_n} * \alpha'.$$

Concatenate by the horizontal map γ_n. As argued above, the concatenation $\gamma_n * \overline{\gamma_n}$ based at p is horizontally null-homotopic, thus we have the following string of homotopic maps:

$$\alpha \simeq \gamma_n * \overline{\gamma_n} * \alpha \simeq \gamma_n * \overline{\gamma_n} * \alpha' \simeq \alpha'.$$

Thus, the map Γ is injective.

Finally, we will show that Γ is surjective. Take a representative β of a homotopy class $[\beta] \in \pi_n^{\mathsf{H}}((M, \xi), p')$. The concatenation $\gamma_n * \beta$ is a horizontal map that is a representative for any element in $\pi_n^{\mathsf{H}}((M, \xi), p)$. Since the concatenation $\overline{\gamma_n} * \gamma_n$ is horizontally homotopic, we have the following string of equalities of elements in

$\pi_n^{\mathsf{H}}((M,\xi),p')$:

$$\Gamma[\gamma_n * \beta] = [\overline{\gamma_n} * \gamma_n * \beta] = [\beta].$$

Therefore, the map Γ is surjective. □

Lipschitz homotopy groups

We now define Lipschitz homotopy groups of a metric space. Per the previous sections of this chapter, we only need to define a pointed sheaf and then take the sheaf's homotopy groups.

Let (M,d) be a metric space. For a manifold N, define the set of maps of sets from N to M such that the map is locally Lipschitz with respect to some Riemannian metric $\mathfrak{g}$ on N:

$$\mathcal{F}_{(M,d)}(N) \coloneqq \{f : N \to M \ : \ \exists \mathfrak{g} \text{ on } N \ni \ f : (N, d_g) \to (M,d) \text{ is locally Lipschitz}\}.$$

Observation 3.0.57. If a map $f : N \longrightarrow M$ is included in $\mathcal{F}_{(M,d)}(N)$ with respect to a Riemannian metric $\mathfrak{g}$, then the map f will be shown to be locally Lipschitz with respect to any Riemannian metric on N. Indeed, let $\mathfrak{g}'$ be another Riemannian metric on N. By Lemma 2.0.60, the identity map

$$\mathbb{1}_N : (N, d_{\mathfrak{g}'}) \longrightarrow (N, d_{\mathfrak{g}})$$

is locally Lipschitz. Thus, the composition of locally Lipschitz maps

$$f = f \circ \mathbb{1}_N : (N, d_{\mathfrak{g}'}) \xrightarrow{\mathbb{1}_N} (N, d_{\mathfrak{g}}) \xrightarrow{f} (M,d)$$

is locally Lipschitz.

Observation 3.0.58. Let $g : N' \longrightarrow N$ be a smooth map between manifolds and let $f : N \longrightarrow M$ be a map of sets contained in the collection $\mathcal{F}_{(M,d)}(N)$. Consider the map of sets given by composition $f \circ g : N' \longrightarrow M$. The map f is locally Lipschitz with respect to some Riemannian metric $\mathfrak{g}$ on N. Let $\mathfrak{g}'$ be any Riemannian metric on N'. By Lemma 2.0.60, since g is a smooth map, it is locally Lipschitz with respect to Riemannian metrics $\mathfrak{g}'$ on N' and $\mathfrak{g}$ on N. Thus, the map

$$f \circ g : (N', \mathfrak{g}') \longrightarrow (M, d)$$

is locally Lipschitz and the map of sets $f \circ g$ is in the collection $\mathcal{F}_{(M,d)}(N')$.

These observations yield that the following is a presheaf:

$$\begin{array}{rcc} \mathcal{F}_{(M,d)} : \mathbf{Man}^{\mathsf{op}} & \to & \mathbf{Set} \\ N & \mapsto & \mathcal{F}_{(M,d)}(N) \\ (N' \xrightarrow{g} N) & \mapsto & \left(\begin{array}{rcl} \mathcal{F}_{(M,d)}(N) & \xrightarrow{g^*} & \mathcal{F}_{(M,d)}(N') \\ f & \mapsto & f \circ g \end{array} \right). \end{array}$$

Notation 3.0.59. Suppose that the metric space (M, d) has a base point p. For a manifold N, denote the subcollection of $\mathcal{F}_{(M,d)}(N)$ for which the maps are based at p by $\mathcal{F}^p_{(M,d)}(N)$.

Lemma 3.0.60. *For a metric space (M, d), the presheaf $\mathcal{F}_{(M,d)} : \boldsymbol{Man}^{\mathsf{op}} \longrightarrow \boldsymbol{Set}$ is a sheaf.*

Proof. Let $N \in \mathbf{Man}$ with two-term open cover $\{U, V\}$. Fix a Riemannian metric $\mathfrak{g}$ on the manifold N. Let $Z \in \mathbf{Set}$ be a set with maps such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[drr, bend left, "f"] \arrow[dr, dotted, "\varphi"] \arrow[ddr, bend right, "g"'] & & \\ & \mathcal{F}_{(M,d)}(N) \arrow[r] \arrow[d] & \mathcal{F}_{(M,d)}(U) \arrow[d] \\ & \mathcal{F}_{(M,d)}(V) \arrow[r] & \mathcal{F}_{(M,d)}(U \cap V). \end{tikzcd}$$

Then, for every element $z \in Z$, there are maps $f(z) : U \to M$ and $g(z) : V \to M$ that are locally Lipschitz with respect to some Riemannian metrics on U and V and that agree on $U \cap V$. By Observation 3.0.57, the map $f(z)$ is locally Lipschitz with respect to $\mathfrak{g}|_U$, the map $g(z)$ is locally Lipschitz with respect to $\mathfrak{g}|_V$, and the map $f(z)|_{U \cap V} = g(z)|_{U \cap V}$ is locally Lipschitz with respect to $\mathfrak{g}|_{U \cap V}$. We then can define a morphism $\varphi : Z \to \mathsf{Map}(N, M)$ into the collection of maps of sets from N to M by

$$\varphi(z)(x) := \begin{cases} f(z)(x) & \text{if} \quad x \in U \\ g(z)(x) & \text{if} \quad x \in V. \end{cases}$$

Since the restriction maps $\varphi(z)|_U$ and $\varphi(z)|_V$ are locally Lipschitz with respect to $\mathfrak{g}$, the map $\varphi(z)$ is locally Lipschitz with respect to $\mathfrak{g}$ on N. So, the map φ takes image in the collection of maps $\mathcal{F}_{(M,d)}(N)$.

Since locally Lipschitz maps are continuous and continuous maps form a sheaf on the metric space M (Observation 3.0.21), the map φ is the unique filler to this diagram. So, the presheaf $\mathcal{F}_{(M,d)}$ satisfies the descent condition for two-term open covers. We will now check that it satisfies the descent condition for sequential covers.

For a manifold N, let $\{U_i\}$ be a sequential open cover of N. Fix a Riemannian metric $\mathfrak{g}$ on the manifold N. Suppose there exists a set Z and maps of sets f_i such that the following solid diagram commutes:

$$\begin{tikzcd}
Z \arrow[d, dashed, "\exists!"'] \arrow[rr, bend left, "f_i"] \arrow[dd, dotted, bend right=60, "f"'] & & & \\
\mathcal{F}_{(M,d)}(N) \arrow[r] \arrow[d, hook] & \dots \arrow[r] & \mathcal{F}_{(M,d)}(U_i) \arrow[r] \arrow[d, hook] & \dots \\
\mathsf{Map}^{\mathsf{cont}}(N, M) \arrow[r] & \dots \arrow[r] & \mathsf{Map}^{\mathsf{cont}}(U_i, M) \arrow[r] & \dots .
\end{tikzcd}$$

As is indicated in the solid diagram, since locally Lipschitz maps are continuous, there are inclusions from each set of locally Lipschitz maps on an open subset to the set of continuous maps on the same open subset. Since the presheaf $\mathsf{Map}^{\mathsf{cont}}(-, M)$ is a sheaf, there exists a unique map f from the set Z to $\mathsf{Map}^{\mathsf{cont}}(N, M)$, indicated by the dotted arrow, such that this diagram commutes.

Now, we will argue that the map of sets f maps into the set of locally Lipschitz maps from the manifold N. For each point $z \in Z$, the map $f(z) : N \to M$ is a continuous map such that, for each index i, the restriction $f(z)|_{U_i}$ agrees with the map $f_i(z)$. By assumption, the map $f_i(z)$ is a locally Lipschitz map from U_i into (M, d) with respect to one, and thus any, Riemannian metric on U_i (Observation 3.0.57). In particular, the map $f_i(z)$ is locally Lipschitz with respect to the Riemannian metric $\mathfrak{g}|_{U_i}$. So, the continuous map $f(z)|_{U_i}$ is locally Lipschitz on U_i with respect to $\mathfrak{g}|_{U_i}$.

Since the open subsets U_i cover the manifold N, the continuous map $f(z)$ is locally Lipschitz on N with respect to the Riemannian metric $\mathfrak{g}$. Thus, the map f factors through $\mathcal{F}_{(M,d)}(N)$, as is indicated by the dashed arrow in the diagram.

Finally, note that f is the unique map that fills the diagram as it is the unique map from Z to $\mathsf{Map}^{\mathsf{cont}}(N, M)$ that allows the diagram to commute. By Proposition 3.0.19, the presheaf $\mathcal{F}_{(M,d)}$ is a sheaf.

□

Now, we should stipulate a natural transformation $\widehat{*} \to \mathcal{F}_{(M,d)}$ to upgrade the sheaf $\mathcal{F}_{(M,d)}$ to a pointed sheaf. Let $p \in M$ be a base point for the metric space

M. Then, $p : * \to (M,d)$ is a constant map and thus locally Lipschitz map into the metric space (M,d). Thus, via Yoneda lemma, there is a natural transformation $\widehat{p} : \widehat{*} \to \mathcal{F}_{(M,d)}$ associated to the point p. Thus, we will take $(\mathcal{F}_{(M,d)}, \widehat{p})$ to be the pointed sheaf associated to (M,d) with base point p.

Definition 3.0.61. For a metric space (M,d) with a base point p, the *nth Lipschitz homotopy group* is

$$\pi_n^{\mathrm{Lip}}((M,d),p) \coloneqq \pi_n(\mathcal{F}_{(M,d)}, \widehat{p}).$$

This definition agrees with the definition of Lipschitz homotopy groups provided in Definition 4.1 in [6]. As is stated in the following lemma, the Lipschitz homotopy groups can be defined in terms of Lipschitz maps from the n-sphere and homotopies with domain $\mathbb{R} \times \mathbb{S}^n$.

Lemma 3.0.62. *For any metric space (M,d) and any point $p \in M$, there is a bijection of sets*

$$\pi_n^{Lip}((M,d),p) \cong \mathsf{Map}_p^{\mathsf{Lip}}(\mathbb{S}^n, M)/\sim,$$

where two based Lipschitz maps in $\mathsf{Map}_p^{\mathsf{Lip}}(\mathbb{S}^n, M)$ are equivalent if there exists a Lipschitz homotopy in $\mathsf{Map}_p^{\mathsf{Lip}}(\mathbb{R} \times \mathbb{S}^n, M)$ between them, that is, $f_0 \sim f_1$ means there is an element $H \in \mathsf{Map}_p^{\mathsf{Lip}}(\mathbb{R} \times \mathbb{S}^n, M)$ for which the restrictions $H|_{\{0\}\times\mathbb{S}^n} = f_0$ and $H|_{\{1\}\times\mathbb{S}^n} = f_1$.

Proof. By a similar argument to Lemma 3.0.53, where the sheaf $\mathcal{F}_{(M,\Delta)}$ is replaced by the sheaf $\mathcal{F}_{(M,d)}$, it is clear that there is a bijection between the sets

$$\pi_n^{\mathrm{Lip}}((M,\Delta),p) \cong \mathcal{F}^p_{(M,d)}(\mathbb{S}^n)/\sim',$$

where two based, locally Lipschitz maps $\alpha, \beta \in \mathcal{F}^p_{(M,d)}(\mathbb{S}^n)$ are equivalent, $\alpha \sim' \beta$, if there exists a map $H \in \mathcal{F}_{(M,d)}(\mathbb{R} \times \mathbb{S}^n)$ such that there is an equality of maps

$H(0,-) = \alpha$ and $H(1,-) = \beta$. Also, the map H must be based at p, that is, $H(-,0) = p$.

Fix the standard Riemannian metrics on $\mathbb{S}^n$ and $\mathbb{R} \times \mathbb{S}^n$. Consider a map $\alpha \in \mathcal{F}^p_{(M,d)}(\mathbb{S}^n)$. By Observation 3.0.57, the map α is locally Lipschitz with respect to the standard metric on $\mathbb{S}^n$. Also, since the n-sphere is compact, the map α is Lipschitz by Lemma 2.0.63. It is immediate that any Lipschitz map on $\mathbb{S}^n$ is locally Lipschitz with respect to some metric on $\mathbb{S}^n$. Thus, we have an equality of sets:

$$\mathcal{F}^p_{(M,d)}(\mathbb{S}^n) = \mathsf{Map}_p^{\mathsf{Lip}}(\mathbb{S}^n, M).$$

Define the map

$$\begin{aligned} \mathsf{Map}_p^{\mathsf{Lip}}(\mathbb{S}^n, M)/\sim' &\to \mathsf{Map}_p^{\mathsf{Lip}}(\mathbb{S}^n, M)/\sim \\ [\alpha]' &\mapsto [\alpha] \end{aligned}$$

where the notation $[-]'$ is meant to indicate equivalence classes with respect to $\sim'$. First, we will argue that this map is well-defined. Take two based Lipschitz maps representing the same equivalence class in the domain $\alpha \sim' \beta$. There exists a map $H : \mathbb{R} \times \mathbb{S}^n \to M$ that is locally Lipschitz with respect to the standard metric on $\mathbb{R} \times \mathbb{S}^n$ (Lemma 3.0.57). Restrict this map to the compact set $[0,1] \times \mathbb{S}^n$. By Lemma 2.0.63, the map $H|_{[0,1]\times\mathbb{S}^n}$ is Lipschitz. Extend this restricted map to all of $\mathbb{R} \times \mathbb{S}^n$ by the assignment:

$$H'(t,p) := \begin{cases} H(0,p) & \text{if } t \le 0 \\ H(t,p) & \text{if } 0 \le t \le 1 \\ H(1,p) & \text{if } t \ge 1. \end{cases}$$

The map H' is Lipschitz and based since $H|_{[0,1]\times\mathbb{S}^n}$ is based and Lipschitz. Thus, the maps $H'(0,-) = \alpha$ and $H'(1,-) = \beta$ are equivalent with respect to $\sim$. Therefore, the map in question is well-defined.

Obviously, the map is surjective as, for any element $[\alpha]$ of the target and any

representative α of the equivalence class, the element of the domain $[\alpha]'$ is mapped to $[\alpha]$. To show injectivity, suppose $\alpha \sim \beta$. Then, there is a Lipschitz homotopy $H : \mathbb{R} \times \mathbb{S}^n \to M$ witnessing the equivalence. The Lipschitz map H is locally Lipschitz with respect to the fixed metric on $\mathbb{R} \times \mathbb{S}^n$. Thus, $H \in \mathcal{F}_{(M,d)}(\mathbb{R} \times \mathbb{S}^n)$ witnesses an equivalence $\alpha \sim' \beta$. So, the map is injective. □

Observation 3.0.63. For a Riemannian manifold (M, g), all smooth maps into M are locally Lipschitz with respect to the path metric d_g on M (Lemma 2.0.60). Thus, for any point $p \in M$, there is an obvious map

$$\pi_n(M, p) \longrightarrow \pi_n^{\mathrm{Lip}}((M, d_g), p).$$

Observation 3.0.64. For a manifold M that has a sub-Riemannian structure (M, ξ, g') and a Riemannian structure (M, g), all maps that are locally Lipschitz with respect to the Carnot-Carathéodory metric are also Lipschitz with respect to the path metric induced by Riemannian metric g (Theorem 2.10, [24]). Thus, for any point $p \in M$, there is a forgetful map

$$\pi_n^{\mathrm{Lip}}((M, d_{CC}^M), p) \longrightarrow \pi_n^{\mathrm{Lip}}((M, d_g), p).$$

Furthermore, all horizontal maps into the manifold endowed with a distribution (M, ξ) are locally Lipschitz with respect to the Carnot-Carathéodory metric induced by the sub-Riemannian metric g' (Lemma 2.0.59). So, there is a forgetful map from horizontal homotopy groups to Lipschitz homotopy groups of a sub-Riemannian manifold. Moreover, from Observation 3.0.55, we have the following diagram of forgetful maps between various homotopy groups:

$$\begin{array}{ccc} \pi_n^{\mathsf{H}}((M,\xi),p) & \longrightarrow & \pi_n^{\mathrm{Lip}}((M,d_{CC}^M),p) \\ \downarrow & & \downarrow \\ \pi_n(M,p) & \longrightarrow & \pi_n^{\mathrm{Lip}}((M,d_g),p). \end{array}$$

The base point will often be suppressed when it is not of utmost importance. In fact, for any contact manifold, as was the case with the horizontal case, the Lipschitz homotopy groups are the same no matter the choice of base point.

Lemma 3.0.65. *For a contact manifold (M,ξ) endowed with a sub-Riemannian structure and two points $p,p' \in M$, the nth Lipschitz homotopy group based at p is isomorphic to the nth Lipschitz homotopy group based at p':*

$$\pi_n^{Lip}((M,d_{CC}^M),p) \cong \pi_n^{Lip}((M,d_{CC}^M),p').$$

Proof. Chow-Rashevskii theorem implies that the metric space (M,d_{CC}^M) is Lipschitz path connected (Theorem 2.2, [24]). Thus, there exists a Lipschitz path $\gamma : I \to (M,d_{CC}^M)$ that connects the point $\gamma(0) = p$ to the point $\gamma(1) = p'$. The construction of a bijection exactly follows the construction in Lemma 3.0.56 where all instances of horizontal maps are replaced with Lipschitz maps.

□

Example 3.0.66. For $n \geq 2$, $\pi_1^{\mathrm{Lip}}(\mathbb{S}^n) = 0$. Indeed, $\mathbb{S}^n$ is simply connected and there is no restriction of Lipschitz maps into $\mathbb{S}^n$ endowed with a Riemannian metric. See Theorem 4.3 in [6].

FIRST HORIZONTAL HOMOTOPY GROUPS OF CONTACT 3-MANIFOLDS

The primary approach for studying contact 3-manifolds in this article is implementing a program of probing the space with horizontal maps. Homotopy groups are employed to report back the results of this probing. Before defining the notion of homotopy group that will be used, we define a notion of homotopy between horizontal maps, called *horizontal homotopy*.

Definition 4.0.1. Let $f, g : N \longrightarrow (M, \xi)$ be horizontal maps from a manifold N into a contact manifold (M, ξ). A horizontal map

$$H : [1,2] \times N \longrightarrow (M, \xi)$$

is a *horizontal homotopy* between f and g if $H|_{\{1\}\times N} = f$ and $H|_{\{2\}\times N} = g$. When such an H exists, f and g are said to be *horizontally homotopic*.

In most texts, a homotopy would have domain $[0,1] \times N$. In this article, the closed interval $[1,2]$ is used instead of $[0,1]$ as this will make an embedding of $[1,2]\times\mathbb{S}^1$ into $\mathbb{R}^2$ more readily apparent. These definitions are obviously equivalent as the two intervals are isomorphic.

We now proceed in defining *horizontal homotopy groups*. This definition is equivalent to the one provided in Definition 4.1 in [6] and in Definition 3.0.52. See Lemma 3.0.53.

Definition 4.0.2. Let (M, ξ) be a contact manifold. The *nth horizontal homotopy group* is

$$\pi_n^{\mathsf{H}}(M, \xi) \coloneqq \mathsf{Map}^{\mathsf{H}}\big(\mathbb{S}^n, (M, \xi)\big) / \sim,$$

where $f \sim g$ for $f, g \in \mathsf{Map}^{\mathsf{H}}\big(\mathbb{S}^n, (M, \xi)\big)$ if f is horizontally homotopic to g.

The main focus of this section is identifying properties of $\pi_1^{\mathsf{H}}(M,\xi)$ for a contact 3-manifold (M,ξ). As will be shown, this group is uncountably generated. Dejarnette et al. proved that this is the case when the contact 3-manifold is the first Heisenberg group [6]. The heart of this result is that horizontal maps, in particular horizontal homotopies, cannot have rank more than 1. Then, for distinct horizontal loops, no horizontal homotopy will be able to traverse the "area" bounded by the curves. Thus, two horizontal paths being horizontally homotopic is seen to be extremely rare.

The logic of the argument will proceed as follows: First, smooth homotopies between distinct smooth embeddings of $\mathbb{S}^1$ into a general manifold M will be considered and revealed to have rank 2 somewhere. Given two such embeddings γ_1 and γ_2, is will be shown that there exists a 1-form on M which

- has compact support on a portion of $\mathrm{Im}(\gamma_2)$,
- pulls back via γ_2 to a 1-form whose integral is non-zero on $\mathbb{S}^1$, and
- pulls back via γ_1 to a 1-form whose integral vanishes on $\mathbb{S}^1$.

Given a smooth homotopy H between γ_1 and γ_2, it will be shown that the pullback of the exterior derivative of this 1-form via H will have non-zero integral over $[1,2]\times\mathbb{S}^1$ and thus H must have rank 2 somewhere in its domain.

Next, properties of contact 3-manifolds will be used to show that $\pi_1^{\mathsf{H}}(M,\xi)$ is an uncountable set. The key observation is that no horizontal map into a contact 3-manifold can have rank greater than 1. Thus, each distinct horizontal embedding of $\mathbb{S}^1$ into (M,ξ) will yield a distinct element in $\pi_1^{\mathsf{H}}(M,\xi)$ as any smooth homotopy between them would need to have rank 2 somewhere. As there are uncountably many horizontal embeddings into $\mathbb{H}^1$, and thus (M,ξ) via Darboux's theorem, $\pi_1^{\mathsf{H}}(M,\xi)$ will be shown to be uncountable.

Finally, with some knowledge of the free group, it will be argued that any uncountable group must be uncountably generated, completing the argument.

Smooth Homotopies Between Distinct Embeddings of $\mathbb{S}^1$

Throughout this section, we will be considering smooth embeddings $\gamma_1, \gamma_2 : \mathbb{S}^1 \hookrightarrow M$ into a manifold M such that

$$\mathrm{Im}(\gamma_1) \neq \mathrm{Im}(\gamma_2).$$

Note that, in this section, we are not concerned with contact manifolds, just properties of smooth embeddings of $\mathbb{S}^1$ and possible smooth homotopies between them. All arguments proceed as if γ_2 is a distinguished embedding. This choice is made arbitrarily and all arguments following identically if the roles of γ_1 and γ_2 are switched.

The goal of this section is to verify that any smooth homotopy between distinct embeddings of $\mathbb{S}^1$ has rank 2 somewhere. In order to do this, a 1-form ω on M will be constructed that has some favorable properties. Namely, that its support lives in an open neighborhood of a portion of $\mathrm{Im}(\gamma_2)$, that its pullback via γ_2 to a 1-form on $\mathbb{S}^1$ has non-zero integral, and that its pullback via γ_1 to a 1-form on $\mathbb{S}^1$ has a vanishing integral. Then, provided such a smooth homotopy H between γ_1 and γ_2 exists, the pullback of $d\omega$ via H will have non-zero integral which will imply that H has rank 2 somewhere.

Before constructing the desired 1-form, it is argued that for distinct smooth embeddings, there is an open subset of $\mathbb{S}^1$ that maps into $\mathrm{Im}(\gamma_2)$ but not $\mathrm{Im}(\gamma_1)$. This open set will be important in the construction of ω.

Such an open set must exist for distinct smooth embeddings of $\mathbb{S}^1$. Otherwise, since the embedded copies of $\mathbb{S}^1$ are not equal as sets, there would be an embedding

of $\mathbb{S}^1$ into a copy of itself, necessitating a diffeomorphism and a contradiction. To verify this claim, the inverse function theorem is used.

Theorem 4.0.3. *(Inverse function theorem, Theorem 2-11 in [30]) Let $f : N \longrightarrow N'$ be a smooth map between manifolds and let $p \in N$. Suppose that the derivative of f at p,*

$$D_p f : T_p N \longrightarrow T_{f(p)} N',$$

is an isomorphism. Then, there exists an open neighborhood $U \subset N$ of p such that

$$f|_U : U \longrightarrow f(U)$$

is a diffeomorphism.

The inverse function theorem will be used to argue that a smooth embedding of a compact manifold into a connected manifold of the same dimension must be a diffeomorphism. In particular, the inverse function theorem makes it apparent that a smooth embedding from one manifold into another of the same dimension is a local diffeomorphism.

Lemma 4.0.4. *Let N be a compact, nonempty manifold and N' be a manifold such that N' and N have the same dimension. Let $\iota : N \hookrightarrow N'$ be a smooth embedding of N into N'. Then ι is a diffeomorphism.*

Proof. It is enough to argue that ι is surjective. As ι is a smooth embedding, it is in particular an immersion. So, for all $p \in N$, the linear map $D_p\iota : T_pN \to T_{\iota(p)}N'$ is injective. As these vector spaces have the same dimension, $D_p\iota$ is an isomorphism for all $p \in N$. By the inverse function theorem, ι is a local diffeomorphism and thus an open map.

Since N is compact, $\iota(N) \subset N'$ is compact and thus closed. $\iota(N) \subset N'$ is also open since ι is an open map. Thus, $\iota(N)$ is closed, open, and nonempty set. Since N' is connected, $\iota(N) = N'$ and ι is surjective. As ι was assumed to be a smooth embedding, this yields that the map is in fact a diffeomorphism. □

With this lemma in hand, it will be shown that there is an open subset of $\mathbb{S}^1$ that is embedded smoothly by γ_2 and γ_1 to distinct sets in M.

Lemma 4.0.5. *Let* $\gamma_1, \gamma_2 : \mathbb{S}^1 \hookrightarrow M$ *be smooth embeddings such that* $\mathrm{Im}(\gamma_2) \neq \mathrm{Im}(\gamma_1)$. *Then,*

$$\gamma_2^{-1}(\mathrm{Im}(\gamma_2) \setminus \mathrm{Im}(\gamma_1)) \subset \mathbb{S}^1$$

is non-empty and open.

Proof. First, it is argued that $\gamma_2^{-1}(\mathrm{Im}(\gamma_2) \setminus \mathrm{Im}(\gamma_1))$ is nonempty. Suppose that this set was empty, that it that $\mathrm{Im}(\gamma_2) \subset \mathrm{Im}(\gamma_1)$. Thus, γ_2 factors smoothly through the submanifold $\mathrm{Im}(\gamma_1)$:

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[rr, hook, "\gamma_2"] \arrow[dr, dashed] & & M. \\
& \mathrm{Im}(\gamma_1) \arrow[ur, hook, "\subset"'] &
\end{tikzcd}$$

So, γ_2 is a smooth embedding of $\mathbb{S}^1$ into the embedded submanifold $\mathrm{Im}(\gamma_1)$.

Now, $\mathbb{S}^1$ is compact. $\mathbb{S}^1$ is also connected and, via the diffeomorphism

$$\gamma_1 : \mathbb{S}^1 \longrightarrow \mathrm{Im}(\gamma_1)$$

the embedded set $\mathrm{Im}(\gamma_1)$ is also connected. Also, the diffeomorphism γ_1 yields that $\mathbb{S}^1$ and $\mathrm{Im}(\gamma_1)$ both have dimension 1. As γ_2 is a smooth embedding of $\mathbb{S}^1$ into $\mathrm{Im}(\gamma_1)$, via Lemma 4.0.4, γ_2 must be a diffeomorphism. In particular, γ_2 maps onto

$\mathrm{Im}(\gamma_1)$. Thus, $\mathrm{Im}(\gamma_1) = \mathrm{Im}(\gamma_2)$, a contradiction. Therefore, $\gamma_2^{-1}(\mathrm{Im}(\gamma_2) \setminus \mathrm{Im}(\gamma_1))$ is nonempty.

Now, to show that this set is open, note that $\mathrm{Im}(\gamma_2) \cap \mathrm{Im}(\gamma_1) \subset \mathrm{Im}(\gamma_2)$ is a closed subset of the submanifold $\mathrm{Im}(\gamma_2)$. Thus, $\mathrm{Im}(\gamma_2) \setminus \mathrm{Im}(\gamma_1)$ is an open subset of $\mathrm{Im}(\gamma_2)$. Therefore, since γ_2 is continuous, $\gamma_2^{-1}(\mathrm{Im}(\gamma_2) \setminus \mathrm{Im}(\gamma_1))$ is open in $\mathbb{S}^1$.

□

It has now been shown that there is an open, nonempty subset of $\mathbb{S}^1$ on which γ_1 and γ_2 map to disjoint subsets of M. As γ_2 is a diffeomorphism mapping from this open set onto its image, a 1-form α will be constructed on this open set that can be pulled back via γ_2^{-1} to a 1-form on $\mathrm{Im}(\gamma_2) \setminus \mathrm{Im}(\gamma_1)$, which will determine ω on this subset of M.

We desire that the support of α lives entirely in the open subset $\gamma_2^{-1}(\mathrm{Im}(\gamma_2) \setminus \mathrm{Im}(\gamma_1)) \subset \mathbb{S}^1$ and that the integral of α over all of $\mathbb{S}^1$ is non-zero. This is to ensure that the to-be-constructed ω pullsback via γ_2 to an integral that is non-zero over $\mathbb{S}^1$.

Before constructing α and verifying that it has the desired properties, recall what is meant by support of a form and partitions of unity subordinate to an open cover.

Definition 4.0.6. Let N be a manifold. For a smooth k-form $\theta \in \Omega^k(N)$, the *support* of θ, denoted $\mathsf{supp}(\theta)$, is the closure of the set $\{p \in N : \theta_p \not\equiv 0\}$.

Theorem 4.0.7. *(Existence of Partitions of Unity [31]) For an open cover $\{U_\beta\}_{\beta \in J}$ of a manifold M, there is a collection of smooth functions, i.e., smooth 0-forms,*

$$\rho_\beta : M \longrightarrow [0,1]$$

such that

1. *The collection of sets* $\{p \in M : \rho_\beta \neq 0\}_{\beta \in J}$ *is locally finite,*

2. $\sum_{\beta \in J} \rho_\beta(p) = 1$ *for all* $p \in M$*, and*

3. *For each* β*, the support of* ρ_β *is contained in* U_β*;*

$$\mathsf{supp}(\rho_\beta) \subset \mathsf{U}_\beta.$$

Such a collection of maps is referred to as a *partition of unity.* The collection of maps $\{\rho_\beta\}_{\beta \in J}$ is said to be *subordinate* to the open cover $\{U_\beta\}_{\beta \in J}$.

Before proceeding to the construction of α, note a useful property partitions of unity in the event that the open cover consists of two terms.

Observation 4.0.8. In the event that $\{\rho_1, \rho_2\}$ is a partition of unity subordinate to a two-term open cover $\{U_1, U_2\}$,

$$\rho_1|_{U_1 \setminus U_2} \equiv 1 \text{ and } \rho_2|_{U_2 \setminus U_1} \equiv 1.$$

Indeed, if $p \in U_1 \setminus U_2$, p is not contained in U_2 and thus not contained in the support of ρ_2. By condition (2) of partitions of unity,

$$\rho_1(p) = \rho_1(p) + \rho_2(p) = 1.$$

The same follows if the roles of 1 and 2 are reversed.

We now proceed with defining a 1-form α on any non-empty open subset of $\mathbb{S}^1$ whose integral over $\mathbb{S}^1$ is non-zero.

The construction of α begins by choosing a coordinate neighborhood contained within V and constructing a partition of unity appropriately. The 1-form α will be one

of these two functions scaling the volume form on $\mathbb{S}^1$. Since the functions associated to partitions of unity are non-negative, the integral of α will be non-zero.

Lemma 4.0.9. *For any nonempty open subset $V \subset \mathbb{S}^1$, there is a 1-form $\alpha \in \Omega^1(\mathbb{S}^1)$ with compact support contained in an open set $U \subset V$ such that $\int_{\mathbb{S}^1} \alpha \neq 0$.*

Proof. Let $p \in V$ and (x, U) be a coordinate neighborhood about p such that $U \subset V$. There exists some $\varepsilon > 0$ such that $x : U \longrightarrow (-\varepsilon, \varepsilon)$ is a diffeomorphism and $x(p) = 0$.

Take a partition of unity subordinate to

$$\{U, \mathbb{S}^1 \setminus x^{-1}[-\varepsilon/2, \varepsilon/2]\},$$

a two-term open cover of $\mathbb{S}^1$. Denote the resulting partition of unity

$$\{\rho_U, \rho_{\mathbb{S}^1 \setminus x^{-1}[-\varepsilon/2, \varepsilon/2]}\}.$$

Finally, let $\mathsf{vol} \in \Omega^1(\mathbb{S}^1)$ be a volume form on $\mathbb{S}^1$. Such a form exists as $\mathbb{S}^1$ is orientable.

Define the smooth 1-form

$$\alpha := \rho_U \ \mathsf{vol} \in \Omega^1(\mathbb{S}^1)$$

and consider $\int_{\mathbb{S}^1} \alpha$. As ρ_U smoothly vanishes outside of U, the support of α is contained within U and

$$\int_{\mathbb{S}^1} \alpha = \int_U \alpha.$$

On U, the 1-form vol can be written in terms of the 1-form dx. More precisely, there exists a smooth function $f : U \longrightarrow \mathbb{R}$ such that

$$\mathsf{vol} = f \ dx.$$

As vol is a volume form, f is non-vanishing. Without loss of generality, assume that f maps into the positive reals.

Thus, $\alpha|_U = (\rho_U \cdot f)\ dx$. As both of the real-valued functions, ρ_U and f, are non-negative, $\rho_U \cdot f$ is non-negative on U. Further, by Observation 4.0.8, $\rho_U \equiv 1$ on $x^{-1}[-\varepsilon/2, \varepsilon/2]$ and

$$\int_U \alpha = \int_U (\rho_U \cdot f)\ dx \geq \int_{x^{-1}[-\varepsilon/2,\varepsilon/2]} f\ dx > 0.$$

This inequality comes from f being a strictly positive function. Thus, $\int_{\mathbb{S}^1} \alpha \neq 0$ and α is a 1-form on $\mathbb{S}^1$ with the desired properties. $\square$

Thus, there exists a 1-form α on $\mathbb{S}^1$ with support in $\gamma_2^{-1}(\mathrm{Im}(\gamma_2) \smallsetminus \mathrm{Im}(\gamma_1))$ whose integral over $\mathbb{S}^1$ is non-zero.

Next, we pullback α via γ_2^{-1} to a 1-form on $\mathrm{Im}(\gamma_2) \smallsetminus \mathrm{Im}(\gamma_1) \subset M$. In order to produce the desired 1-form ω on M, we will first extend $(\gamma_2^{-1})^*\alpha$ to a 1-form on an open neighborhood of $\mathrm{Im}(\gamma_2) \smallsetminus \mathrm{Im}(\gamma_1)$ via tubular neighborhood theorem before expanding to all of M via a partition of unity.

Before constructing ω, recall the tubular neighborhood theorem.

Theorem 4.0.10. *(Tubular Neighborhood Theorem) Let $X \subset M$ be a submanifold of a manifold M. Then, there is an open set $\nu \subset M$ such that $X \subset \nu$ and there is a smooth map*

$$\pi : \nu \longrightarrow X$$

such that $\pi|_X = \mathbb{1}_X$.

Lemma 4.0.11. *Let $\gamma_1, \gamma_2 : \mathbb{S}^1 \hookrightarrow M$ be smooth embeddings of $\mathbb{S}^1$ into a manifold M*

such that $\mathrm{Im}(\gamma_1) \neq \mathrm{Im}(\gamma_2)$*. Then, there is a smooth 1-form* $\omega \in \Omega^1(M)$ *such that*

$$\int_{\mathbb{S}^1} \gamma_1^* \omega = 0 \ \textit{and} \ \int_{\mathbb{S}^1} \gamma_2^* \omega \neq 0.$$

Proof. First, note that there is a 1-form α on $\mathbb{S}^1$ that has nonzero integral and has support inside $\gamma_2^{-1}(\mathrm{Im}(\gamma_2) \smallsetminus \mathrm{Im}(\gamma_1))$. Indeed, by Lemma 4.0.5 and Lemma 4.0.9, there is a nonempty open set $U \subset \gamma_2^{-1}(\mathrm{Im}(\gamma_2) \smallsetminus \mathrm{Im}(\gamma_1))$ and a 1-form $\alpha \in \Omega^1(\mathbb{S}^1)$ such that α has compact support $\mathsf{supp}(\alpha) \subset \mathsf{U}$ and non-vanishing integral $\int_U \alpha = \int_{\mathbb{S}^1} \alpha \neq 0$.

Consider the 1-form $(\gamma_2^{-1})^* \alpha|_U \in \Omega^1(\gamma_2(U))$ on the submanifold $\gamma_2(U) \subset M$. It will be extended to a 1-form ω on the entirety of M such that its support is contained inside a tubular neighborhood of $\gamma_2(U)$. In particular, ω will vanish on $(\mathrm{Im}(\gamma_1) \cup \mathrm{Im}(\gamma_2)) \smallsetminus \gamma_2(U)$

Begin by extending $(\gamma_2^{-1})^* \alpha|_U$ to a tubular neighborhood of $\gamma_2(U)$. This tubular neighborhood is chosen so as to only intersect the images of γ_1 and γ_2 in $\gamma_2(U)$. Indeed,

$$(\mathrm{Im}(\gamma_1) \cup \mathrm{Im}(\gamma_2)) \smallsetminus \gamma_2(U)$$

is a closed subset of M. Thus, a tubular neighborhood ν about $\gamma_2(U)$ can be found in the complement of this closed set in M:

$$\gamma_2(U) \subset \nu \subset M \smallsetminus ((\mathrm{Im}(\gamma_1) \cup \mathrm{Im}(\gamma_2)) \smallsetminus \gamma_2(U)).$$

Also, there exists a smooth map $\pi : \nu \longrightarrow \gamma_2(U)$ such that $\pi|_{\gamma_2(U)} = \mathbb{1}_{\gamma_2(U)}$. Note that $\nu \cap (\mathrm{Im}(\gamma_1) \cup \mathrm{Im}(\gamma_2)) = \gamma_2(U)$ and $\gamma_2^{-1}(\nu) = U$.

Extend $(\gamma_2^{-1})^* \alpha|_U$ to ν by pullback with respect to π:

$$(\gamma_2^{-1} \circ \pi)^* \alpha|_U = \pi^* (\gamma_2^{-1})^* \alpha|_U \in \Omega^1(\nu).$$

So, the initial 1-form $(\gamma_2^{-1})^*\alpha|_U$ defined on the submanifold $\gamma_2(U)$ of M is extended to a 1-form on a tubular neighborhood ν of the original submanifold.

We will now extend the 1-form $(\gamma_2^{-1})^*\alpha|_U$ from $\gamma_2(U)$ to all of M, by scaling $(\gamma_2^{-1}\circ\pi)^*\alpha|_U$ via a partition of unity. Take a partition of unity subordinate to the two-term open cover $\{\nu, M\smallsetminus\gamma_2(\mathsf{supp}(\alpha))\}$. Denote these maps ρ_ν and $\rho_{M\smallsetminus\gamma_2(\mathsf{supp}(\alpha))}$ respectively. Finally, finish the extension of $(\gamma_2^{-1})^*\alpha|_U$ to all of M by defining $\omega\in\Omega^1(M)$ as follows:

$$\omega := \begin{cases} \rho_\nu\cdot(\gamma_2^{-1}\circ\pi)^*\alpha|_U & \text{on } \nu \\ 0 & \text{on } M\smallsetminus\nu \end{cases}$$

Since ρ_ν smoothly vanishes outside of ν, this is indeed a smooth 1-form on the entirety of M.

Now, ω will be shown to satisfy the desired integral properties of the Lemma.

First, it will be shown that $\int_{\mathbb{S}^1}\gamma_1^*\omega = 0$. It will be argued that $\gamma_1^*\omega$ is identically zero on all of $\mathbb{S}^1$ and, therefore, its integral vanishes.

Take $p\in\mathbb{S}^1$. Then, since $\mathrm{Im}(\gamma_1)\cap\nu=\varnothing$, it follows that $\gamma_1(p)\in M\smallsetminus\nu$. Thus, the linear functional $\omega_{\gamma_1(p)}: T_{\gamma_1(p)}M\longrightarrow\mathbb{R}$ is the zero map by definition of ω and

$$(\gamma_1^*\omega)_p = \omega_{\gamma_1(p)}\circ D_p\gamma_1 = 0.$$

Since p was arbitrary, $\gamma_1^*\omega\equiv 0$ and its integral vanishes; $\int_{\mathbb{S}^1}\gamma_1^*\omega = 0$.

Now it will be shown that $\int_{\mathbb{S}^1}\gamma_2^*\omega\neq 0$. Use that $\mathbb{S}^1=\gamma_2^{-1}(\nu)\coprod\gamma_2^{-1}(M\smallsetminus\nu)$ and $U=\gamma_2^{-1}(\nu)$ to write

$$\int_{\mathbb{S}^1}\gamma_2^*\omega = \int_U\gamma_2^*\omega + \int_{\gamma_2^{-1}(M\smallsetminus\nu)}\gamma_2^*\omega.$$

This integral over $\gamma_2^{-1}(M\smallsetminus\nu)$ will vanish as $\gamma_2^*\omega\equiv 0$. Indeed, since ω is defined to be 0 on $M\smallsetminus\nu$, its pullback to $\gamma_2^{-1}(M\smallsetminus\nu)$ via γ_2 must be identically zero. So, the

right-most integral vanishes; $\int_{\gamma_2^{-1}(M\smallsetminus\nu)} \gamma_2^*\omega = 0$.

Now it will be shown that $\int_U \gamma_2^*\omega \neq 0$. Let $p \in U$. Then, using the definition of ω, the linear functional

$$(\gamma_2^*\omega)_p : T_p(\mathbb{S}^1) \longrightarrow \mathbb{R}$$

can be expressed as follows:

$$\begin{aligned}
(\gamma_2^*\omega)_p &= \omega_{\gamma_2(p)} \circ D_p\gamma_2 \\
&= \rho_\nu(\gamma_2(p)) \cdot [(\gamma_2^{-1} \circ \pi)^*\alpha|_U]_{\gamma_2(p)} \circ D_p\gamma_2 \\
&= \rho_\nu(\gamma_2(p)) \cdot \alpha|_{U\ \gamma_2^{-1}\circ\pi\circ\gamma_2(p)} \circ D_{\pi\circ\gamma_2(p)}\gamma_2^{-1} \circ D_{\gamma_2(p)}\pi \circ D_p\gamma_2 \\
&= \rho_\nu(\gamma_2(p)) \cdot \alpha|_{U\ \gamma_2^{-1}\circ\pi\circ\gamma_2(p)} \circ D_p(\gamma_2^{-1} \circ \pi \circ \gamma_2).
\end{aligned}$$

The last equality in this string is do to the Chain Rule.

Since $\gamma_2(p) \in \gamma_2(U)$ and π restricted to $\gamma_2(U)$ is the identity, $\pi \circ \gamma_2(p) = \gamma_2(p)$. Since $p \in U$ is arbitrary, on the entirety of the open set U,

$$\gamma_2^{-1} \circ \pi \circ \gamma_2|_U = \mathbb{1}_U.$$

Thus,

$$\begin{aligned}
(\gamma_2^*\omega)_p &= \rho_\nu(\gamma_2(p)) \cdot \alpha|_{U\ \gamma_2^{-1}\circ\pi\circ\gamma_2(p)} \circ D_p(\gamma_2^{-1} \circ \pi \circ \gamma_2) \\
&= \rho_\nu(\gamma_2(p)) \cdot \alpha|_{U\ p} \circ \mathbb{1}_{T_p(\mathbb{S}^1)} \\
&= \rho_\nu(\gamma_2(p)) \cdot \alpha|_{U\ p},
\end{aligned}$$

and $\gamma_2^*\omega = (\rho_\nu \circ \gamma_2) \cdot \alpha|_U$ on U.

Since the partition of unity was taken subordinate to a two-term open cover, by

Observation 4.0.8, $\rho_\nu \equiv 1$ on $\gamma_2(\mathsf{supp}(\alpha))$. So, on $\mathsf{supp}(\alpha)$,

$$\gamma_2^*\omega = (\rho_\nu \circ \gamma_2) \cdot \alpha|_U = \alpha|_U.$$

On $U \smallsetminus \mathsf{supp}(\alpha)$, since $\mathsf{supp}(\alpha)$ is the support of $\alpha|_U$,

$$\gamma_2^*\omega = (\rho_\nu \circ \gamma_2) \cdot \alpha|_U \equiv 0 \equiv \alpha|_U$$

Thus, $\gamma_2^*\omega = \alpha|_U$ on all of U and, by the choice of α,

$$\int_{\mathbb{S}^1} \gamma_2^*\omega = \int_U \gamma_2^*\omega = \int_U \alpha \neq 0.$$

Therefore, the constructed 1-form ω on M has all desired properties.

$\square$

The 1-form ω will be used to show that any smooth homotopy between distinct embeddings of $\mathbb{S}^1$ into a manifold must have rank 2 somewhere. The procedure for verifying this claim will be to pullback the 2-form $d\omega$ via such a homotopy which will integrate to a non-zero value. The following lemma verifies that a non-vanishing integral of the pullback of a 2-form via a smooth homotopy is enough to guarantee that the homotopy has rank 2 somewhere.

Lemma 4.0.12. *Let $n \geq 2$. Let N be a n-dimensional compact manifold (possibly with boundary), $H : N \longrightarrow M$ be a smooth map into a manifold M, and let $\theta \in \Omega^n(M)$ be a smooth n-form on M. If $\int_N H^*\theta \neq 0$, then H has rank n on a nonempty open set.*

Proof. First, it will be shown that the n-form $H^*\theta$ cannot be identically zero on the entirety of N. Then, it will be argued that H must have rank n everywhere that

$H^*\theta \not\equiv 0$. Thus, H will have rank n on a non-empty set.

Begin by naming the subset of N on which $H^*\theta$ is not identically zero. Define the subset $A \subset N$ as

$$A := \{p \in N : (H^*\theta)_p \not\equiv 0\}.$$

The set A will be shown to be an open subset of N, and thus a submanifold. The n-form $H^*\theta$ on N is a smooth section of the nth exterior power of the cotangent bundle of N:

$$\begin{tikzcd} & \Lambda^n(T^*N) \arrow[d] \\ N \arrow[ur, "H^*\theta"] \arrow[r, "\mathbb{1}_N"] & N. \end{tikzcd}$$

The zero section of $\Lambda^n(T^*N) \to N$ is an embedded copy of N in $\Lambda^n(T^*N)$:

$$\zeta : N \hookrightarrow \Lambda^n(T^*N)$$

$$p \longmapsto 0_p,$$

where 0_p is the alternating n-tensor on T_pN that is identically zero. Thus, $\zeta(N) \subset \Lambda^n(T^*N)$ is closed and

$$A = (H^*\theta)^{-1}(\Lambda^n(T^*N) \setminus \zeta(N))$$

is open in N. Thus, A and $N \setminus A$ are measurable with respect to Lebesgue measure pulled back from any local coordinates.

The set A will be shown to be nonempty. Proceeding by contradiction, suppose that A is empty. The integral $\int_N H^*\theta$ can be written as

$$\int_N H^*\theta = \int_A H^*\theta + \int_{N\setminus A} H^*\theta = \int_{N\setminus A} H^*\theta.$$

The integral over A vanishes in the case that $A = \varnothing$. Since $H^*\theta \equiv 0$ on $N \setminus A$, the integral of $H^*\theta$ over $N \setminus A$ is 0. Thus, $\int_N H^*\theta = 0$, contradicting the hypothesis.

Now that it has been shown that A is non-empty, it will be shown that H must have rank n on the entirety of A. Proceeding by contradiction, suppose that the rank of H was strictly less than n for some point $p \in A$. As will be shown, this implies that $(H^*\theta)_p \equiv 0$.

Let $v_1, \dots, v_n \in T_pN$. Since the rank of D_pH is strictly less than n, the image of these vectors under the map D_pH cannot span $T_{H(p)}M$. So, there exists a vector in the above list v_j such that

$$D_pHv_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} c_i \cdot D_pHv_i$$

for scalars $c_i \in \mathbb{R}$. Thus,

$$\begin{aligned}
(H^*\theta)_p(v_1, \dots, v_j, \dots, v_n) &= \theta_{H(p)}(D_pHv_1, \dots, D_pHv_j, \dots, D_pHv_n) \\
&= \theta_{H(p)}(D_pHv_1, \dots, \sum_{\substack{i=1 \\ i \neq j}}^{n} c_i \cdot D_pHv_i, \dots, D_pHv_n) \\
&= \sum_{\substack{i=1 \\ i \neq j}}^{n} c_i \cdot \theta_{H(p)}(D_pHv_1, \dots, D_pHv_i, \dots, D_pHv_i, \dots, D_pHv_n) \\
&= 0.
\end{aligned}$$

This last equality follows from $\theta_{H(p)}$ being an alternating n-tensor and the vector D_pHv_i appearing twice as input in each term of the summation.

As the vectors $v_1, \dots, v_n$ were arbitrary, it follows that $(H^*\theta)_p \equiv 0$. This contradicts that p is an element in A. Thus, H has rank n on the nonempty open set A.

□

For the purposes of this chapter, we aim to look at the rank of a smooth

homotopy H between distinct smooth embeddings γ_1 and γ_2 of $\mathbb{S}^1$. The 2-form that will be pulled back is $d\omega$, where ω is the 1-form established in Lemma 4.0.11. In order to integrate $H^*d\omega$, Stokes' theorem will be of great use.

Theorem 4.0.13. *(Stokes' Theorem, see Chapter 8, Theorem 4 in [31]) Let N be an n-dimensional oriented manifold with boundary and let $\omega \in \Omega^{n-1}(N)$ be a smooth $(n-1)$-form with compact support on N. The two real numbers*

$$\int_N d\omega = \int_{\partial N} \omega$$

are identical.

Remark 4.0.14. Later on in the article, when our attention is turned to the Lipschitz case rather than the horizontal one, the existence of a Lipschitz version of Stokes' theorem will be of importance.

We now proceed with the main result of this subsection: verifying that smooth homotopies between embeddings of $\mathbb{S}^1$ with distinct images must have rank 2 somewhere on their domain.

Lemma 4.0.15. *Let $\gamma_1, \gamma_2 : \mathbb{S}^1 \hookrightarrow M$ be smooth embeddings of $\mathbb{S}^1$ into a manifold M such that $\mathrm{Im}(\gamma_1) \neq \mathrm{Im}(\gamma_2)$. Let $H : [1,2] \times \mathbb{S}^1 \longrightarrow M$ be a smooth homotopy between γ_1 and γ_2. Then H has rank 2 on a non-empty set.*

Proof. We will show that there is a 2-form on M such that the integral of its pullback via H is nonzero. This is enough to guarantee that H has rank 2 on a nonempty set via Lemma 4.0.12.

Let $\omega \in \Omega^1(M)$ be the 1-form constructed in Lemma 4.0.11. The exterior derivative of ω yields the desired 2-form $d\omega$ on M.

With the 2-form $d\omega$ chosen, we will argue that $\int_{[1,2]\times\mathbb{S}^1} H^* d\omega$ is nonzero. This will complete the proof as Lemma 4.0.12 yields that H must have rank 2 somewhere.

As pullback and exterior derivative commute, we have an equality of 2-forms: $H^* d\omega = d\ H^*\omega$. Via an application of Stokes' theorem, there are equalities among real numbers:

$$\int_{[1,2]\times\mathbb{S}^1} H^*\ d\omega = \int_{[1,2]\times\mathbb{S}^1} d\ H^*\omega = \int_{\{1\}\times\mathbb{S}^1 \amalg \{2\}\times\mathbb{S}^1} H^*\omega = -\int_{\{1\}\times\mathbb{S}^1} H^*\omega + \int_{\{2\}\times\mathbb{S}^1} H^*\omega.$$

Thus, only the integrals of $H^*\omega$ over $\{1\}\times\mathbb{S}^1$ and $\{2\}\times\mathbb{S}^1$ need to be considered.

Since $H|_{\{i\}\times\mathbb{S}^1} = \gamma_i$ for $i = 1,2$, these integrals can be written in terms of the original embeddings, and their integrals are already known via Lemma 4.0.11:

$$\int_{\{1\}\times\mathbb{S}^1} H^*\omega = \int_{\{1\}\times\mathbb{S}^1} \gamma_1^*\omega = 0 \text{ and } \int_{\{2\}\times\mathbb{S}^1} H^*\omega = \int_{\{2\}\times\mathbb{S}^1} \gamma_2^*\omega \neq 0.$$

Therefore, putting this all together, we have shown that

$$\int_{[1,2]\times\mathbb{S}^1} H^*\ d\omega = -\int_{\{1\}\times\mathbb{S}^1} \gamma_1^*\omega + \int_{\{2\}\times\mathbb{S}^1} \gamma_2^*\omega \neq 0.$$

Therefore, by Lemma 4.0.12, H has rank 2 on a nonempty open set.

□

$\pi_1^{\mathsf{H}}(M,\xi)$ is an Uncountable Set

Having shown that smooth homotopies between distinct smooth embeddings of $\mathbb{S}^1$ must have rank 2 on a non-empty subset, we return our attention to the objects of primary interest in this article, contact 3-manifolds.

To reach the goal of this subsection, first, Lemma 4.0.15 will be used in conjunction with a key observation about horizontal maps into a contact 3-manifold.

The key observation will be that all such horizontal maps have rank at most 1. This will prevent any horizontal homotopy from existing between distinct horizontal embeddings of $\mathbb{S}^1$ into a contact 3-manifold.

Thus, each distinct horizontal embedding of $\mathbb{S}^1$ into a contact 3-manifold (M, ξ) (i.e., Legendrian knot) will be shown to determine a distinct equivalence class in $\pi_1^{\mathsf{H}}(M, \xi)$. So, a lower bound on the cardinality of $\pi_1^{\mathsf{H}}(M, \xi)$ will be found by considering how many Legendrian knots there are in (M, ξ). For our purposes, it will be enough to gain a lower bound on the cardinality of Legendrian knots in the first Heisenberg group $\mathbb{H}^1$. Since all contact 3-manifolds are locally modeled on $\mathbb{H}^1$ by Darboux's Theorem, there are at least as many Legendrian knots in (M, ξ) as there are in $\mathbb{H}^1$. Using tools outlined in Example 2.0.37, it will be shown that there are an uncountable number of Legendrian knots in $\mathbb{H}^1$.

Corollary 4.0.16. *Let $\gamma_1, \gamma_2 : \mathbb{S}^1 \hookrightarrow (M, \xi)$ be horizontal embeddings of $\mathbb{S}^1$ into a 3-dimensional contact manifold (M, ξ) such that $\mathrm{Im}(\gamma_1) \neq \mathrm{Im}(\gamma_2)$. Then γ_1 and γ_2 are not horizontally homotopic.*

Proof. Suppose that such a horizontal homotopy H exists. By Lemma 4.0.15, H has rank 2 on a nonempty set. But, by the key observation (Lemma 2.0.47), horizontal maps into a contact 3-manifold have rank at most 1 everywhere, a contradiction.

□

Corollary 4.0.16 will be reworded as the existence of an injective map of sets with target $\pi_1^{\mathsf{H}}(M, \xi)$. This vantage point will make it easier to begin discussing the cardinality of $\pi_1^{\mathsf{H}}(M, \xi)$.

Before defining this injective map and naming its domain, we define what is meant by a Legendrian submanifold and a Legendrian knot in a contact manifold.

Definition 4.0.17. Let (M, ξ) be a contact manifold. A *Legendrian submanifold* in (M, ξ) is the image of a horizontal embedding of a manifold N into (M, ξ);

$$\iota : N \hookrightarrow (M, \xi).$$

Remark 4.0.18. The key observation is consistent with dimensions of Legendrian submanifolds in contact manifolds. For a contact $(2n+1)$-manifold, it is known that the largest dimension a Legendrian submanifold can have is n [13]. For contact 3-manifolds, a Legendrian submanifold is the result of a horizontal embedding ι of a manifold into a contact 3-manifold. By the key observation, ι can have rank at most 1. Thus, the dimension of a horizontally embedded submanifold in to a contact 3-manifold is bounded by 1, which is consistent with prior results.

We now define Legendrian knots, which are Legendrian submanifolds where the embedded submanifold is diffeomorphic to $\mathbb{S}^1$.

Definition 4.0.19. Let (M, ξ) be a contact manifold. A *Legendrian knot* in (M, ξ) is the image of a horizontal embedding of $\mathbb{S}^1$ into (M, ξ):

$$\gamma : \mathbb{S}^1 \hookrightarrow (M, \xi).$$

We will say that $K = \mathrm{Im}(\gamma)$ is a Legendrian knot and that γ paramterizes the knot K.

Before producing the desired injective map, recall the Hopf degree theorem. This result will be used to show that the useful injective map is well-defined.

Theorem 4.0.20. *(The Hopf degree theorem; Chapter 3, Section 6 in [16]) Let N be a compact, connected, oriented n-manifold and take smooth maps $f, g : N \longrightarrow \mathbb{S}^n$. Then, f and g are smoothly homotopic if and only if they have the same degree.*

Throughout this article, the base point of any manifold (contact or otherwise) will be suppressed. Also, fix the standard positive orientation on $\mathbb{S}^1$.

Corollary 4.0.21. *There is an injective map of sets*

$$\left\{ \begin{array}{c} \text{Based at } p \text{, Oriented,} \\ \text{Legendrian Knots in } (M,\xi) \end{array} \right\} \hookrightarrow \pi_1^{\mathsf{H}}(M,\xi)$$

$$K \longmapsto [\gamma]$$

where $\gamma : \mathbb{S}^1 \hookrightarrow (M,\xi)$ *is a based, orientation-preserving embedding parametrizing the knot* K*.*

Proof. First, the indicated map will be shown to be well-defined. Let K be a based, oriented, Legendrian knot in (M,ξ). By definition, there is at least one $\gamma : \mathbb{S}^1 \hookrightarrow (M,\xi)$ parametrizing the knot K.

Suppose that $\gamma, \psi : \mathbb{S}^1 \hookrightarrow (M,\xi)$ are horizontal embeddings parametrizing the same knot K, that is that $\mathrm{Im}(\gamma) = K = \mathrm{Im}(\psi)$. Further, assume each parametrization agrees with the orientation on K. To verify that this map is well-defined, it will be shown that γ and ψ are horizontally homotopic.

As ψ maps diffeomorphically onto the submanifold K,

$$\psi^{-1} : K \longrightarrow \mathbb{S}^1$$

is a smooth, orientation-preserving diffeomorphism. Further, as the composition of orientation-preserving maps is orientation-preserving, the map $\psi^{-1} \circ \gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ also orienation-preserving. Thus, we have the following diagram, where each arrow is an orientation-preserving diffeomorphism:

$$\begin{tikzcd} \mathbb{S}^1 \arrow[dd, dashed, "\psi^{-1}\circ\gamma"', "\cong"] \arrow[dr, "\gamma", "\cong"'] & \\ & K. \\ \mathbb{S}^1 \arrow[ur, "\cong", "\psi"'] & \end{tikzcd}$$

As $\psi^{-1} \circ \gamma$ is an orientation-preserving automorphism of $\mathbb{S}^1$, it is smoothly homotopic to the identity map on $\mathbb{S}^1$. Indeed, the degree of any orientation-preserving diffeomorphism is +1. The Hopf degree theorem guarantees that the self-maps $\psi^{-1} \circ \gamma$ and $\mathbb{1}_{\mathbb{S}^1}$ of $\mathbb{S}^1$ are smoothly homotopic. Let

$$h : [1,2] \times \mathbb{S}^1 \longrightarrow \mathbb{S}^1$$

be a smooth homotopy witnessing $\psi^{-1} \circ \gamma$ homotopic to $\mathbb{1}_{\mathbb{S}^1}$. Then,

$$\psi \circ h : [1,2] \times \mathbb{S}^1 \longrightarrow (M, \xi)$$

is a horizontal homotopy between $\gamma = \psi \circ (\psi^{-1} \circ \gamma)$ and $\psi = \psi \circ \mathbb{1}_{\mathbb{S}^1}$. The smooth homotopy $\psi \circ h$ is indeed horizontal as post-composing a smooth map h by a horizontal map ψ yields a horizontal map $\psi \circ h$ (Lemma 2.0.8).

So, orientation-preserving maps γ and ψ paramterizing the knot K are horizontally homotopic. Since γ and ψ were arbitrary orientation-preserving parametrizations of K, the same horizontal homotopy class of $\pi_1^{\mathsf{H}}(M, \xi)$ is determined no matter the choice of orientation-preserving parametrization of a knot K. Thus, the map is well-defined.

Now, the map will be shown to be injective. Take distinct, based, oriented, Legendrian knots $K \neq K'$ in (M, ξ). Take horizontal embeddings $\gamma, \gamma' : \mathbb{S}^1 \hookrightarrow (M, \xi)$ parametrizing K and K', respectively. Then, $\mathrm{Im}(\gamma) = K \neq K' = \mathrm{Im}(\gamma')$.

By Corollary 4.0.16, each parametrization determines a distinct horizontal homotopy classes in $\pi_1^{\mathsf{H}}(M,\xi)$ as γ is not horizontally homotopic to γ'. Thus, distinct, based, oriented, Legendrian knots in (M,ξ) determine distinct horizontal homotopy classes in $\pi_1^{\mathsf{H}}(M,\xi)$ and the map is injective. □

To show that there are uncountably many based, oriented, Legendrian knots in any contact 3-manifold, it is enough to show that there are uncountably many based, oriented, Legendrian knots in $\mathbb{H}^1$. Indeed, the Theorem of Darboux (Theorem 2.0.36) indicates that there is a distributional embedding of $\mathbb{H}^1$ centered at the base point in a contact 3-manifold.

By Lemma 2.0.12, any Legendrian knot based at 0 in $\mathbb{H}^1$ is carried by this distributional embedding to a Legendrian knot in the contact 3-manifold with the fixed base point. It will be shown that there are uncountably many Legendrian knots in $\mathbb{H}^1$ via the tools used in Example 2.0.37. This will establish the following result.

Lemma 4.0.22. *For any contact 3-manifold (M,ξ) and any choice of base point, there are uncountably many based, oriented, Legendrian knots.*

Proof. Let $p \in M$ denote the base point. First, it will be shown that there is an injective map of sets from Legendrian knots in $\mathbb{H}^1$ based at 0 to Legendrian knots in (M,ξ) based at p.

By the Theorem of Darboux, there is a distributional embedding $\varphi : \mathbb{H}^1 \hookrightarrow (M,\xi)$ such that $\varphi(0) = p$. Distributional maps send horizontal mappings to horizontal mappings (Lemma 2.0.12). Additionally, since φ is a smooth embedding, φ maps Legendrian knots in $\mathbb{H}^1$ to Legendrian knots in (M,ξ). Thus, there is a map of sets,

$$\left\{ \begin{array}{c} \text{Based at 0, Oriented,} \\ \text{Legendrian Knots in } \mathbb{H}^1 \end{array} \right\} \hookrightarrow \left\{ \begin{array}{c} \text{Based at } p\text{, Oriented,} \\ \text{Legendrian Knots in } (M,\xi) \end{array} \right\}$$

$$K \longmapsto \varphi(K).$$

This map will be shown to be injective. Since φ is a smooth embedding, there is a smooth inverse defined on its image: $\varphi^{-1} : \varphi(\mathbb{H}^1) \to \mathbb{H}^1$. Thus, for Legendrian knots K and K' in $\mathbb{H}^1$, if their images under φ are equal in (M,ξ), that is $\varphi(K) = \varphi(K')$, then

$$K = \varphi^{-1} \circ \varphi(K) = \varphi^{-1} \circ \varphi(K') = K'.$$

Therefore, the defined map is injective.

Thus, in order to show that there are uncountably many based, oriented, Legendrian knots in (M,ξ), it is enough to show that there are uncountably many based at 0, oriented, Legendrian knots in $\mathbb{H}^1$. An uncountable family of such Legendrian knots will be generated to verify this claim.

It is known that horizontal paths in $\mathbb{H}^1$ are completely determined by their projections into the xy-plane (Example 2.0.37). Namely, if

$$(\gamma_1,\gamma_2) : [0,2\pi] \longrightarrow \mathbb{R}^2$$

is a smooth path in $\mathbb{R}^2$, and an initial z-coordinate $\gamma_3(0)$ is specified, there is a unique way to determine $\gamma_3 : [0,2\pi] \longrightarrow \mathbb{R}$ such that $\gamma = (\gamma_1,\gamma_2,\gamma_3)$ is a horizontal path in $\mathbb{H}^1$:

$$\gamma_3(t) = \gamma_3(0) + 2\int_0^t \gamma_1'(s)\gamma_2(s) - \gamma_2'(s)\gamma_1(s)\ ds. \tag{$*$}$$

In order to guarantee that γ is a loop based at 0, set $\gamma_3(0) = 0$ and demand that

$$\int_0^{2\pi} \gamma_1'(s)\gamma_2(s) - \gamma_2'(s)\gamma_1(s)\ ds = 0;$$

that is that (γ_1, γ_2) bounds zero signed area.

A family of such loops are given by

$$(\gamma_1^a, \gamma_2^a) : [0, 2\pi] \longrightarrow \mathbb{R}^2$$

$$s \longmapsto (a \sin s, -a \sin 2s)$$

for $a > 0$. Since

$$\frac{d^n\gamma_i^a}{ds^n}(0) = \frac{d^n\gamma_i^a}{ds^n}(2\pi) \text{ for } i = 1, 2,$$

(γ_1^a, γ_2^a) yields a smooth map with domain $\mathbb{S}^1$. Thus, for each $a > 0$, there is a horizontal loop

$$\gamma^a = (\gamma_1^a, \gamma_2^a, \gamma_3^a) : \mathbb{S}^1 \longrightarrow \mathbb{H}^1$$

where γ_3^a is determined by $(*)$.

Let $a > 0$ be fixed. γ^a will be argued to be an embedding of $\mathbb{S}^1$. To see that the map is injective, note that the projection is injective everywhere but $(\gamma_1^a, \gamma_2^a)(0) = (\gamma_1^a, \gamma_2^a)(\pi)$. But, via $(*)$,

$$\gamma_3^a(\pi) = -\frac{8}{3}a^2 \neq 0 = \gamma_3^a(0).$$

So, $\gamma^a(0) \neq \gamma^a(\pi)$ and, more generally, $\gamma^a(s) \neq \gamma^a(s')$ for all distinct s and s' in $\mathbb{S}^1$.

Also, γ^a will be argued to be an immersion. Note that the derivative is never zero, even for the projection (γ_1^a, γ_2^a). Indeed,

$$\left(\frac{d\gamma_1^a}{ds}, \frac{d\gamma_2^a}{ds}\right)(s) = (a \cos s, -2a \cos 2s)$$

and $\frac{d\gamma_1^a}{ds}(s)$ only vanishes when $s = \frac{\pi}{2}, \frac{3\pi}{2}$. But, $\frac{d\gamma_2^a}{ds}\left(\frac{\pi}{2}\right) \neq 0 \neq \frac{d\gamma_2^a}{ds}\left(\frac{3\pi}{2}\right)$. Thus, $D_s\gamma^a$ is never the zero map for any $s \in \mathbb{S}^1$ and, since $\mathbb{S}^1$ is one dimensional, γ^a is an immersion.

So, $\gamma^a(\mathbb{S}^1)$ is a Legendrian knot based at 0 in $\mathbb{H}^1$ that inherits an orientation from the smooth embedding γ^a.

Now, it will be noted that, for distinct positive real numbers $a \neq b$, that $\gamma^a \neq \gamma^b$. Indeed, the point $(a, 0) = (\gamma_1^a, \gamma_2^a)(\frac{\pi}{2})$ appears nowhere on the loop (γ_1^b, γ_2^b) in $\mathbb{R}^2$. As their projections are distinct, the knots $\gamma^a(\mathbb{S}^1)$ and $\gamma^b(\mathbb{S}^1)$ are distinct.

Therefore, since the set $(0, \infty)$ is uncountable, the family of Legendrian knots $\{\gamma^a\}_{a \in (0,\infty)}$ is uncountable. Thus, there are uncountably many based, oriented, Legendrian knots in $\mathbb{H}^1$.

As was shown at the beginning of this argument, there is an injective map from based, oriented, Legendrian knots in $\mathbb{H}^1$ to based, oriented, Legendrian knots in (M, ξ). Since the domain of this injective map is uncountable, so is its target. Thus, there are uncountably many based, oriented, Legendrian knots in (M, ξ) for any base point.

□

Corollary 4.0.23. *For any contact 3-manifold, $\pi_1^{\mathsf{H}}(M, \xi)$ is an uncountable set.*

Proof. By Lemma 4.0.21, there is an injective map into $\pi_1^{\mathsf{H}}(M, \xi)$ whose domain is the collection of all based, oriented, Legendrian knots in (M, ξ). By Lemma 4.0.22, the domain of this injective map is an uncountable set. Thus, the target of this map, $\pi_1^{\mathsf{H}}(M, \xi)$, is uncountable. □

The sheaf $\mathcal{F}_{(M,\xi)}$ is not representable

Corollary 4.0.24. *For a contact 3-manifold* (M,ξ)*, the associated sheaf* $\mathcal{F}_{(M,\xi)}$ *is not representable, i.e., there exists no manifold* N *such that*

$$\mathcal{F}_{(M,\xi)} = \hat{N}.$$

Proof. Suppose that there was a manifold N such that its representable sheaf equals the sheaf $\mathcal{F}_{(M,\xi)}$. By Lemma 3.0.36, there is a bijection between the sets $\pi_1(N)$ and $\pi_1\left(\hat{N}\right)$. Thus, there is a bijective correspondence between the sets $\pi_1^{\mathsf{H}}(M,\xi)$ and $\pi_1(N)$.

By Corollary 4.0.23, the first horizontal homotopy group $\pi_1^{\mathsf{H}}(M,\xi)$ is an uncountable set. But, the fundamental group $\pi_1(N)$ is a countable set (see, for instance, Theorem 8.11 in [20]). Therefore, no such manifold N exists. □

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$\pi_1^{\mathsf{H}}(M,\xi)$ is Uncountably Generated

It will now be argued that $\pi_1^{\mathsf{H}}(M,\xi)$ is uncountably generated. This will follow directly from $\pi_1^{\mathsf{H}}(M,\xi)$ being an uncountable set.

Before making concrete that $\pi_1^{\mathsf{H}}(M,\xi)$ is uncountably generated, the free group on a set of generators will be discussed. The cardinality of a free group will be related to the cardinality of its generating set. In particular, a countable set will yield a countable free group. Given a generating set of a general group, by a universal property of free groups, there will be a surjective map from the free group on the set of generators onto the group. Thus, a connection will be made between cardinality of a group with the cardinality of its generating set. Namely, if a group is uncountable,

then any generating set of the group is also uncountable.

Let S be a set. Associate to S an *alphabet*

$$A(S) := S \coprod S^{-1} \coprod \{1\},$$

where $S^{-1} := \{s^{-1} : s \in S\}$ is the set of formal inverses of elements of S. The *free group* on S, denoted $F(S)$, is the set of reduced words of $A(S)$. Each reduced word can be denoted by a finite string of elements from the alphabet: $s_1 s_2 \dots s_k \in F(S)$ where $s_i \in A(S)$ for each $1 \leq i \leq k$. The *length* of the reduced word is the integer k denoting the length of the string. The free group also contains the empty word, the word of length 0, which is denoted by 1.

For any set S, the associated free group $F(S)$ can be partitioned by the length of words. For any positive integer k, collect all *k-length words*:

$$W^k(S) := \{s_1 s_2 \dots s_k \in F(S) : s_i \neq 1 \text{ for all } 1 \leq i \leq k\}.$$

Further, define that $W^0(S)$ to be the set containing a single element, the 0-length word 1. Thus, $F(S)$ can be written as a countable union of all such collections:

$$F(S) = \coprod_{k=0}^{\infty} W^k(S).$$

There is a natural injective map from the set of k-length words to the k-fold product of the alphabet $A(S)$:

$$\begin{aligned} W^k(S) &\hookrightarrow A(S)^k \\ s_1 s_2 \dots s_k &\longmapsto (s_1, s_2, \dots, s_k) \end{aligned}$$

Here, the 0-fold product $A(S)^0$ is understood to be a singleton set.

Thus, via the universal property of coproducts and the equality of sets $F(S) = \coprod_{k=0}^{\infty} W^k(S)$, there is an injective map

$$F(S) \hookrightarrow \bigcup_{k=0}^{\infty} A(S)^k.$$

So, the cardinality of $F(S)$ depends on the cardinality of the alphabet associated to S, which in turn depends on the cardinality of the original set S.

In the following statements, a set is *countable* if it has finite or countably-infinite cardinality. A set is *uncountable* if it is not countable.

Lemma 4.0.25. *If S is a countable set, then $F(S)$ is a countable set.*

Proof. Since S is countable, so is S^{-1} as there is an obvious bijection between the two sets. Then, $A(S)$ is a 3-fold union of countable sets, and thus countable. For each non-negative integer k, the set $A(S)^k$ is a finite product of countable sets and so is also countable. Therefore, the countable union of countable sets $\cup_{k=0}^{\infty} A(S)^k$ is countable. Since there is an injective map of sets from $F(S)$ into $\cup_{k=0}^{\infty} A(S)^k$, the free group $F(S)$ is countable as well. $\square$

The free group is indeed a group, as is discussed in full in [8]. In particular, the free group has a universal property: For a set S, a group G, and a map of sets $S \longrightarrow G$, there is a unique homomorphism $\pi : F(S) \longrightarrow G$ extending the original map:

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr] & F(S) \arrow[d, dashed, "\exists!\,\pi"] \\ & G. \end{tikzcd}$$

Note that the map on top is the natural inclusion map of a set S into the free group it generates, in particular, by seeing each element of S as a word of length in $W^1(S)$:

$$\begin{aligned} S &\hookrightarrow F(S) \\ s &\longmapsto s. \end{aligned}$$

This property guarantees that for any group G, there is a surjective map from a free group generated by a subset of the group, as is discussed in the following observation.

Observation 4.0.26. For any group G, there exists a subset $S \subset G$ such that the unique homomorphism $\pi : F(S) \longrightarrow G$ which extends the inclusion map $S \hookrightarrow G$ is surjective:

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr] & F(S) \arrow[d, dashed, two heads, "\exists!" left, "\pi"] \\ & G \end{tikzcd}$$

Such a set S is called the *generating set* of G. Obviously, for any group G, such a generating set exists, as the set S can be taken to be the underlying set of the group, $S = G$.

It is then of interest if any smaller generating sets exist. G is called *countably generated* if it has a generating set that is countable. G is called *uncountably generated* if it is not countably generated.

Lemma 4.0.27. *If a group G is countably generated, then G is countable. Thus, if G is uncountable, it must be uncountably generated.*

Proof. Let $S \subset G$ be the generating set for G. By assumption, S is countable. By Lemma 4.0.25, the free group $F(S)$ is countable. By Observation 4.0.26, there is a surjective map $\pi : F(S) \twoheadrightarrow G$. Since $F(S)$ is countable, its image $\pi(F(S)) = G$ is also countable.

$\square$

The main goal of this section is now shown.

Theorem 4.0.28. *For any contact 3-manifold* (M, ξ)*, the group* $\pi_1^{\mathsf{H}}(M, \xi)$ *is uncountably generated.*

Proof. By Lemma 4.0.22 and Corollary 4.0.21, $\pi_1^{\mathsf{H}}(M, \xi)$ contains an uncountable set and is thus uncountable. By Lemma 4.0.27, $\pi_1^{\mathsf{H}}(M, \xi)$ is uncountably generated. $\square$

HIGHER HORIZONTAL HOMOTOPY GROUPS OF CONTACT 3-MANIFOLDS

In this chapter, we investigate an approach for calculating the higher horizontal homotopy groups of a contact 3-manifold. It is conjectured at the end of this chapter that $\pi_n^{\mathsf{H}}(M,\xi)$ is trivial for $n \geq 2$ for any contact 3-manifold (M,ξ). The approach is inspired by the key observation in Lemma 2.0.47, which guarantees that any horizontal map $f : \mathbb{S}^n \longrightarrow (M,\xi)$ has rank 1. Intuitively, this says that the image of such a map f crushes $\mathbb{S}^n$ to a connected collection of (immersed) lines and points.

Rather than focusing on the contact structure of (M,ξ), we investigate smooth maps out of the n-sphere that have rank at most 1. It will be shown that all such maps are smoothly null-homotopic. This yields that any horizontal map into a contact 3-manifold is smoothly null-homotopic, but fails to guarantee that a horizontal null-homotopy exists.

For the remainder of this chapter, fix an $n \geq 2$, a smooth manifold M, and a smooth map

$$f : \mathbb{S}^n \longrightarrow M$$

such that f has rank at most 1. If the rank of the map f is 0 at all points of $\mathbb{S}^n$, the map is constant and thus obviously smoothly null-homotopic. So, the map f can be assumed to be non-constant and is rank 1 somewhere in the n-sphere. Furthermore, fix a Riemannian metric on $\mathbb{S}^n$.

A partition of $\mathbb{S}^n$ will be constructed such that each part of the partition is connected and the map f is constant on each part. This partition will be defined by considering where the rank of the map f is 0 and where it is 1. The set of points in $\mathbb{S}^n$ where the rank of f is 1 will be shown to be an open set with a canonical integrable distribution of codimension 1. The map f will be constant on each leaf of the associated foliation and the collection of compact leafs will be an open subset

that has a codimension 1 foliation. Moreover, in each component of the collection of compact leafs, all leafs will be shown to be diffeomorphic to each other. Using the Jordan-Brouwer separation theorem, each of these leafs separate the space $\mathbb{S}^n$. This will be enough to show that the quotient space associated to the partition is a tree. Since the map f is constant on each part of the partition, it will continuously factor through this tree and thus be smoothy null-homotopic.

The subsets $\mathcal{U}$ and $\mathcal{K}$

Consider the non-constant rank 1 map $f : \mathbb{S}^n \longrightarrow M$. Define sets

$$\mathcal{U} := \{p \in \mathbb{S}^n \mid dim(\ker(D_p f)) = n - 1\}$$

and

$$\mathcal{K} := \mathbb{S}^n \setminus \mathcal{U} = \{p \in \mathbb{S}^n \mid D_p f \equiv 0\}.$$

Lemma 5.0.1. *$\mathcal{K}$ is closed and $\mathcal{U}$ is open in $\mathbb{S}^n$.*

Proof. It will be shown that $\mathcal{K}$ is the continuous preimage of the zero section of a Hom-bundle. Thus, $\mathcal{K}$ will be closed and its complement $\mathcal{U}$ will be open.

First, we specify the Hom-bundle. Pullback the tangent bundle $TM \longrightarrow M$ along the smooth map f to get the smooth vector bundle,

$$f^*TM \longrightarrow \mathbb{S}^n.$$

Along with the tangent bundle $T\mathbb{S}^n \longrightarrow \mathbb{S}^n$, consider the Hom-bundle between $T\mathbb{S}^n$ and f^*TM over $\mathbb{S}^n$. Denote this smooth vector bundle

$$\mathsf{Hom}(T\mathbb{S}^n, f^*TM) \longrightarrow \mathbb{S}^n.$$

It will be argued that Df is a section of this vector bundle. Indeed, by the universal property of pullbacks, the derivative map Df determines a bundle map over $\mathbb{S}^n$:

$$\begin{tikzcd}
T\mathbb{S}^n \arrow[dr, dashed] \arrow[drr, bend left, "Df"] \arrow[ddr, bend right] & & \\
& f^*TM \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & TM \arrow[d] \\
& \mathbb{S}^n \arrow[r, "f"] & M.
\end{tikzcd}$$

Abusing notation, the section of the Hom-bundle that is determined by Df will also be referred to as Df:

$$\begin{array}{ccc}
\mathbb{S}^n & \xrightarrow{Df} & \mathsf{Hom}(T\mathbb{S}^n, f^*TM) \\
 & & \\
p & \longmapsto & \left(\begin{array}{ccc} T_p\mathbb{S}^n & \xrightarrow{D_pf} & (f^*TM)_p \\ (p,v) & \longmapsto & (p, D_pf(v)) \end{array} \right).
\end{array}$$

Denote the zero section of $\mathsf{Hom}(T\mathbb{S}^n, f^*TM) \longrightarrow \mathbb{S}^n$ by

$$\zeta : \mathbb{S}^n \hookrightarrow \mathsf{Hom}(T\mathbb{S}^n, f^*TM)$$

For each point $p \in \mathbb{S}^n$, the element $\zeta(p)$ is the zero map from the vector space $T_p\mathbb{S}^n$ to the vector space $(f^*TM)_p$. The image of the zero section is a closed subset of $\mathsf{Hom}(T\mathbb{S}^n, f^*TM)$ and, since the section Df is continuous, the preimage of the zero section

$$\mathcal{K} = \{p \in \mathbb{S}^n \mid D_pf \equiv 0\} = (Df)^{-1}(\zeta(\mathbb{S}^n))$$

is closed. Since the set $\mathcal{U}$ is the complement to the closed set $\mathcal{K}$ in $\mathbb{S}^n$, the set $\mathcal{U}$ is open in $\mathbb{S}^n$. $\square$

A canonical foliation of $\mathcal{U}$

The open set $\mathcal{U}$ is by definition the collection of all points in $\mathbb{S}^n$ such that the rank of the map f is 1. As such, for all points $p \in \mathcal{U}$, the dimension of the kernel of the linear map D_pf, denoted $\ker(D_pf)$, is $n-1$. Thus, since the dimension of this vector subspace of $T_p\mathbb{S}^n$ is fixed for all points $p \in \mathbb{S}^n$, the kernel of Df forms a codimension 1 distribution over the set $\mathcal{U}$,

$$\ker(Df) \longrightarrow \mathcal{U}.$$

Lemma 5.0.2. *The codimension 1 distribution* $\ker(Df)$ *over the open set* $\mathcal{U}$ *is integrable.*

Proof. Let V be an open set contained in the open set $\mathcal{U}$. Let X and Y be vector fields defined on V such that the vector fields are tangent to the distribution $\ker(Df)$, that is

$$X, Y \in \Gamma(V, \ker(Df)|_V).$$

Since the vector fields X and Y belong to $\ker(Df)$, the zero vector field on M is f-related to the vector fields X and Y. Thus, the Lie bracket of X and Y is f-related to the zero vector field (See, for instance, Proposition 14.17 in [32]). That is, for any point $p \in V$,

$$D_pf\left([X_p, Y_p]\right) = [D_pf(X_p), D_pf(Y_p)] = [0_p, 0_p] = 0_p.$$

So, the Lie bracket of the vectors X_p and Y_p belongs to the distribution, $[X_p, Y_p] = 0_p \in \ker(D_pf)$. Since p and V were arbitrary, the distribution $\ker(Df)$ is closed under Lie brackets, that is, the distribution $\ker(Df)$ is integrable. $\square$

Frobenius integrability theorem states that any integrable distribution has an associated foliation. Thus, the open set $\mathcal{U}$ has a foliation by codimension 1 immersed

submanifolds associated to the integrable distribution $\ker(Df)$. As will be verified, the leafs of this foliation should be thought of as connected level sets of the restricted map $f|_{\mathcal{U}}$.

Remark 5.0.3. For a general foliation, we cannot guarantee that all leafs of the foliation are embedded submanifolds, e.g., the 2-torus foliated by lines with a fixed irrational slope. But, as will be verified, each compact leaf of the foliation associated to the distribution $\ker(Df)$ is a submanifold of $\mathcal{U}$. Indeed, take a compact leaf of the associated foliation. That is, the leaf is an immersed submanifold in $\mathcal{U}$ that is the image of a smooth immersion

$$\iota : L \longrightarrow \mathcal{U}$$

for some compact $(n-1)$-manifold L. Immersed submanifolds are the images of injective immersions. So, the immersion ι is injective as well. Since L is compact, the map ι is also a proper map. Therefore, since ι is an injective and proper immersion, the map ι is an embedding and $\iota(L)$ is a submanifold in $\mathcal{U}$.

Terminology 5.0.4. Unless otherwise noted, a *leaf* of a foliation will refer to the image of the immersion in the foliated manifold rather than the immersion into the foliated manifold along with the domain of this immersion. Unless it is of use, we will dispense with referring to the immersion at all.

The map f is constant on leafs of $\mathcal{U}$ and on components of $\mathcal{K}$

Proposition 5.0.5. *Let $L \subset \mathcal{U}$ be a leaf of the foliation associated to $(\mathcal{U}, \ker(Df))$. Then, the map $f|_L : L \to M$ is constant.*

Proof. Take a point $p \in L$. Take an open and connected neighborhood U of p in the leaf L. Choose a point $q \in U$. Let $\psi : [0,1] \to U$ be any smooth path in U joining p

and q. Post-composing the path ψ by the given map f yields a smooth path in $\mathcal{U}$:

$$f \circ \psi : [0,1] \longrightarrow \mathcal{U}.$$

It will be argued that this path is constant.

Since U is a subset of a leaf of the foliation associated to $\ker(Df)$, at each point $p' \in U$, the tangent space of U agrees with the distribution:

$$T_{p'}U = \ker(D_{p'}f).$$

As ψ is a smooth map into U, its derivative $D\psi$ maps into $\ker(Df)$. So, by the chain rule, the linear map $D(f \circ \psi) = Df \circ D\psi$ must be the zero map:

$$\begin{tikzcd}
 & \ker(Df|) \arrow[r, dashed] \arrow[d, "="] & \mathcal{U} \times \{0\} \arrow[d, hook] \\
 & TU \arrow[r, "Df|"] \arrow[dd] & T\mathcal{U} \arrow[dd] \\
T[0,1] \arrow[uur] \arrow[ur, "D\psi"] \arrow[urr, bend right, "D(f\circ\psi)"'] \arrow[dd] & & \\
 & U \arrow[r, "f|"] & \mathcal{U}. \\
[0,1] \arrow[ur, "\psi"] \arrow[urr, bend right, "f\circ\psi"'] & &
\end{tikzcd}$$

It follows that the path $f \circ \psi : [0,1] \to \mathcal{U}$ is constant. In particular, we have an equality of points

$$f(p) = f \circ \psi(0) = f \circ \psi(1) = f(q).$$

Since q was an arbitrary point in U, the map f is constant on the open subset U. Thus, for each point $p \in L$, there exists an open neighborhood of p such that the map

f restricted to the neighborhood is constant. Since p was an arbitrary point in L, the map $f|_L$ is locally constant. Since the leaf L is connected, the map $f|_L$ is constant.

□

On each connected component of $\mathcal{K}$, the map f is constant. This will be shown in the general setting by Sard's Theorem.

Proposition 5.0.6. *Let $f : N \longrightarrow M^m$ be a smooth map between manifolds. Then, f is constant on the connected components of*

$$\mathcal{K} = \{p \in N \mid D_p f = 0\}.$$

Proof. Fix $K_0 \subset \mathcal{K}$ to be a connected component of $\mathcal{K}$ and take an element $p \in K_0$. We will first show that the statement is correct when $M = \mathbb{R}$ via Sard's theorem. Then, we will show that the statement holds when $M = \mathbb{R}^m$ by using coordinate projection and that the result holds for when the target is $\mathbb{R}$. Finally, we show that the statement is correct when M is any manifold using the local Euclidean structure.

First, assume that the target of the map f is $M = \mathbb{R}$. Then, the derivative map $D_p f$ is not surjective exactly when it is the zero map, $D_p f = 0$. So, $\mathcal{K}$ is the set of critical points of the map f. By Sard's Theorem, the image of $\mathcal{K}$ by the map f has measure zero in $\mathbb{R}$. Thus, for the component $K_0 \subset \mathcal{K}$, the set $f(K_0) \subset f(\mathcal{K})$ has measure zero as well.

Since K_0 is connected and f is continuous, the image $f(K_0)$ is a connected subset of $\mathbb{R}$. The only connected, measure zero subsets of $\mathbb{R}$ are singleton sets. Thus, $f(K_0)$ consists of a single point, i.e., the map f is constant on K_0.

Now, assume that the target of the map f is $M = \mathbb{R}^m$. Consider the projection map

$$\mathsf{pr}_i : \mathbb{R}^m \longrightarrow \mathbb{R}$$

onto the ith coordinate of $\mathbb{R}^m$. Then, the composition $\mathsf{pr}_i \circ f : N \longrightarrow \mathbb{R}$ is a smooth, real-valued map and, by chain rule, we have an equality of linear maps,

$$D_p(\mathsf{pr}_i \circ f) = D_{f(p)}(\mathsf{pr}_i) \cdot D_p f = 0.$$

Thus, p is a critical point of the map $\mathsf{pr}_i \circ f$ and the component K_0 is contained in the set of critical points of the map $\mathsf{pr}_i \circ f$.

By the first argument, the map $\mathsf{pr}_i \circ f$ is constant on the component K_0. By arbitrary choice of i, all coordinates of the image of K_0 by the map f are fixed. Thus, the map f is constant on K_0.

Finally, proceed with the general result. We will use that the manifold M is locally Euclidean and the established results to show that the restricted map $f|_{K_0}$ is locally constant. Since the component K_0 is connected and $f|_{K_0}$ is continuous, the map $f|_{K_0}$ will be constant.

Take a coordinate chart (U, φ) about the point $f(p) \in M$, that is there is a diffeomorphism

$$\varphi : U \xrightarrow{\cong} \varphi(U) \subset \mathbb{R}^n$$

for an open neighborhood U of $f(p)$.

Consider the open set $V = f^{-1}(U)$. The composition $\varphi \circ f|_V : V \longrightarrow \mathbb{R}^m$ is a smooth map defined on the open neighborhood V containing the point p.

For any point $p' \in \mathcal{K} \cap V$, by chain rule, there is an equality of linear maps,

$$D_{p'}(\varphi \circ f) = D_{f(p')}\varphi \cdot D_{p'} f = 0.$$

So, the sets $\mathcal{K} \cap V$ and $K_0 \cap V$ are contained in the set where the derivative of $\varphi \circ f$ vanishes.

Now, there is a non-empty intersection of K_0 and V since both contain the point p. But, the set $K_0 \cap V$ need not be connected. Take the connected component $K_0' \subset K_0 \cap V$ containing the point p. By the preceding argument, the map $\varphi \circ f|_{K_0'}$ is constant. Since the diffeomorphism φ is injective, the map $f|_{K_0'}$ is constant.

To ensure that the map $f|_{K_0}$ is locally constant, it is enough to argue that the set $K_0' \subset K_0$ is open with respect to the subspace topology. Since K_0' is a component of $K_0 \cap V$, it is open in the set $K_0 \cap V$. It is immediate that $K_0 \cap V$ is open in K_0. Thus, K_0' is an open set in K_0.

Therefore, for any point $p \in K_0$, there is an open set $K_0' \subset K_0$ on which $f|K_0$ is constant. Thus, the map $f|K_0$ is locally constant for the connected set K_0 and therefore constant.

□

Compact leafs of $\mathcal{U}$

The map f has been shown to be constant on the leafs of the foliation of $\mathcal{U}$ and on the connected components of $\mathcal{K}$. Though the collection of leafs of $\mathcal{U}$ and the connected components of $\mathcal{K}$ form a partition of $\mathbb{S}^n$ such that the map f is constant on each part, this is not the correct partition. The correct partition, as will be argued, is the partition given by connected components of the level sets of the map f.

Recall that we want to form a partition of $\mathbb{S}^n$ such that the map f is constant on each part and the quotient space associated to this partition is a tree.

Example 5.0.7. Suppose that the map $f : \mathbb{S}^2 \longrightarrow \mathbb{R}$ is the projection of the 2-sphere (with the embedding into $\mathbb{R}^3$ indicated in Figure 5.1) into $\mathbb{R}$ as is shown. It is clear that there are three distinct parts of the partition that are sent to 0 which form a figure eight. Thus, there will not be a means of separating them.

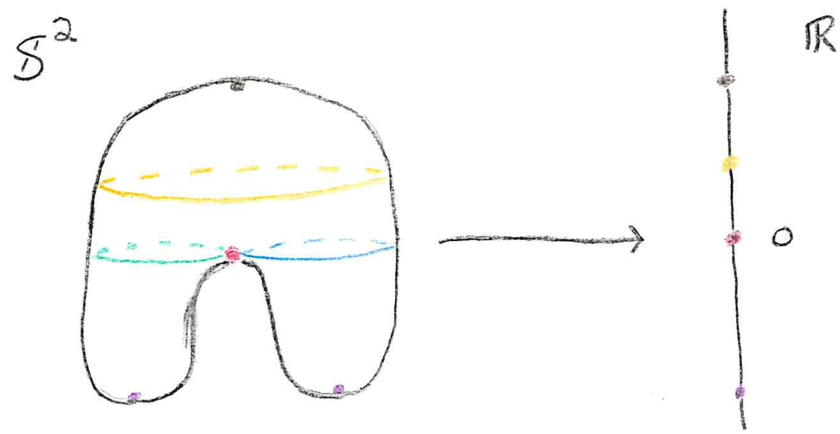

Figure 5.1: The map $f : \mathbb{S}^2 \longrightarrow \mathbb{R}$.

Rather, we want these three parts to be viewed as one part of the partition. To achieve this desire, we will define a partition where parts are *compact* leafs of the foliation of $\mathcal{U}$ and connected components of the complement of the collection of compact leafs.

The map f will be shown to be still constant on each part. The benefit of this approach will be that each compact leaf is codimension 1 in the n-sphere and thus separates the space. This will be used to show that the quotient space associated to the partition is a tree.

Let $\mathcal{U}' \subset \mathcal{U}$ denote the set of points in the open set $\mathcal{U}$ that are contained in compact leafs of the foliation. It will be shown that the set $\mathcal{U}'$ is open in $\mathbb{S}^n$ by constructing a vector field about any compact leaf that respects the foliation; that is, flowing the compact leaf by this vector field recovers all of the leafs nearby. Since flowing a subspace for any time is a diffeomorphism, all of these leafs must be compact.

Before constructing this vector field, some local properties of this foliation are revealed. Of primary importance is that each point of the open set $\mathcal{U}$ has a foliated

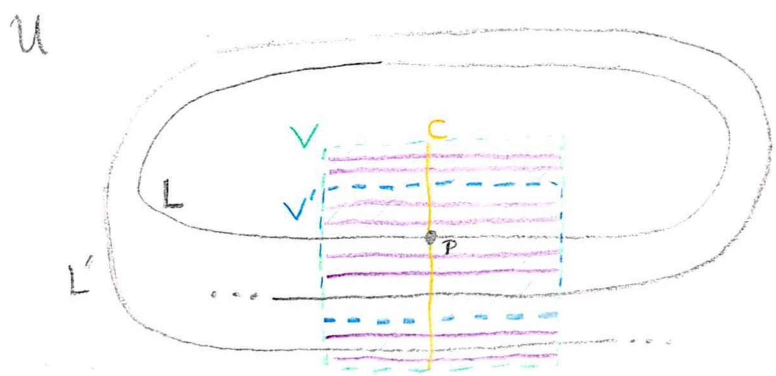

Figure 5.2: Foliated neighborhood of p in the leaf L.

neighborhood on which each leaf maps onto a distinct point in the target M.

Local properties of leafs in $\mathcal{U}$

Proposition 5.0.8. *Let $L \subset \mathcal{U}$ be a leaf of the foliation of the open set $\mathcal{U}$ and $p \in L$ be a point in the leaf. Then, there exists a foliated open neighborhood $V' \subset \mathcal{U}$ of p such that, for any leaf $L' \subset \mathcal{U}$, the leaf L' intersects the neighborhood at most once, i.e., $L' \cap V'$ is either empty or connected.*

Proof. Let $L \subset \mathcal{U}$ be a leaf of the foliation determined by $(\mathcal{U}, \ker(Df))$ and take a point $p \in L$. Since $\mathcal{U}$ has a codimension 1 foliation, there is an open neighborhood $V \subset \mathcal{U}$ of p that has a foliational parametrization

$$x : V \xrightarrow{\cong} (-\varepsilon, \varepsilon)^n$$

such that $x(p) = (0, \dots, 0)$. Also, for any leaf $L' \subset \mathcal{U}$ that has non-empty intersection with V, each component of $L' \cap V$ is sent diffeomorphically via the map x onto $\{\delta\} \times (-\varepsilon, \varepsilon)^{n-1}$ for some $|\delta| < \varepsilon$.

Define a smooth path

$$c : (-\varepsilon, \varepsilon) \longrightarrow V$$

by $c(t) \coloneqq x^{-1}(0,t)$. Consider the smooth path $f \circ c : (-\varepsilon, \varepsilon) \longrightarrow M$. We will show that this is an immersion, and thus a local embedding near 0. This will follow from the path c being transverse to the leafs of the kernel of the map f.

Let $t \in (-\varepsilon, \varepsilon)$. Denote the zero element of the vector space $T_{f \circ c(t)} M$ by $0_{f \circ c(t)}$. We will show that $D_t(f \circ c)^{-1}\{0_{f \circ c(t)}\} = 0_t$ where 0_t is the zero element in the vector space $T_t(-\varepsilon, \varepsilon)$. This will show that $f \circ c$ is an immersion.

By chain rule, we have an equality of linear maps $D_t(f \circ c) = D_{c(t)} f \circ D_t c$. Thus, there is an equality of sets,

$$D_t(f \circ c)^{-1}\{0_{f \circ c(t)}\} = D_t c^{-1}\left(D_{c(t)} f^{-1}\{0_{f \circ c(t)}\}\right) = D_t c^{-1}\left(\ker(D_{c(t)} f)\right).$$

So, in order to show that the map $f \circ c$ is an immersion, it is enough to show the equality of sets,

$$D_t c^{-1}\left(\ker(D_{c(t)} f)\right) = 0_t.$$

Consider the point $x \circ c(t) = (t, 0 \ldots, 0)$. By using the canonical identifications of $T_t(-\varepsilon, \varepsilon) \cong \mathbb{R}$ and $T_{(t,0,\ldots,0)}(-\varepsilon, \varepsilon)^n \cong \mathbb{R}^n$, we identify the linear map $D_t(x \circ c)$ with the linear inclusion into the first coordinate of $\mathbb{R}^n$. So, there is the equality of sets

$$(D_t(x \circ c))^{-1}\left(\{0_t\} \times T_{(0,\ldots,0)}(-\varepsilon, \varepsilon)^{n-1}\right) = 0_t.$$

As the map x is a foliational parametrization of the open set V with respect to the integrable distribution $\ker(Df)$, it is a distributional diffeomorphism and its

derivative map is an isomorphism between vector subspaces,

$$D_{c(t)}x : \ker\left(D_{c(t)}f\right) \xrightarrow{\cong} \{0_t\} \times T_0(-\varepsilon,\varepsilon)^{n-1}.$$

Also, by applying chain rule to the path $c = x^{-1} \circ (x \circ c)$, we have an equality of linear maps,

$$D_t c = \left(D_{c(t)}x\right)^{-1} \circ D_t(x \circ c).$$

Therefore, stringing these equalities together, we have the desired equality of sets:

$$\begin{aligned} D_t c^{-1}\left(\ker(D_{c(t)}f)\right) &= D_t(x \circ c)^{-1}\left(D_{c(t)}x\left(\ker\left(D_{c(t)}f\right)\right)\right) \\ &= D_t(x \circ c)^{-1}(\{0_t\} \times T_0(-\varepsilon,\varepsilon)^{n-1}) \\ &= 0_t. \end{aligned}$$

So, for any time $t \in (-\varepsilon,\varepsilon)$, the map $f \circ c$ is an immersion, in particular at $t = 0$. Thus, as immersions are local embeddings, there exists $\delta > 0$ such that the restricted map

$$f \circ c|_{(-\delta,\delta)} : (-\delta,\delta) \longrightarrow M$$

is an embedding.

Let $V' \subset V$ be the subset of the foliational neighborhood such that the diffeomorphism x restricts to V' in the following way:

$$x|_{V'} : V' \xrightarrow{\cong} (-\delta,\delta) \times (-\varepsilon,\varepsilon)^{n-1}.$$

Thus, the composition of smooth maps

$$\begin{tikzcd}
(-\delta,\delta) \arrow[r, hook, "(\mathbb{1}_{(-\delta,\delta)},0\ldots,0)"] \arrow[rrr, bend right, "f\circ c|_{(-\delta,\delta)}"'] & (-\delta,\delta)\times(-\varepsilon,\varepsilon)^{n-1} \arrow[r, "x|_{V'}^{-1}"] & V' \arrow[r, hook, "f"] & M
\end{tikzcd}$$

is an embedding, and thus injective.

Suppose that the leaf $L' \subset \mathcal{U}$ has non-empty intersection with the open set V'. Proceeding by contradiction, assume that there is at least two components of the intersection $L' \cap V'$. Label these two components as $(L' \cap V')_1$ and $(L' \cap V')_2$.

The map $x|_{V'}$ takes components of leafs in V' to horizontal strips in the product $(-\delta,\delta)\times(-\varepsilon,\varepsilon)^{n-1}$. So, the specified components are brought to distinct strips by the map $x|_{V'}$:

$$x|_{V'}(L' \cap V')_1 = \{\delta_1\} \times (-\varepsilon,\varepsilon)^{n-1}$$

and

$$x|_{V'}(L' \cap V')_2 = \{\delta_2\} \times (-\varepsilon,\varepsilon)^{n-1}$$

for distinct values $\delta_1, \delta_2 \in (-\delta,\delta)$.

Since $f \circ c|$ is injective, the following elements in M are not equal:

$$f(x^{-1}(\delta_1, 0\ldots,0)) = f\circ c(\delta_1) \neq f\circ c(\delta_2) = f(x^{-1}(\delta_2, 0\ldots,0)).$$

But, the points $x^{-1}(\delta_1, 0\ldots,0), x^{-1}(\delta_2, 0\ldots,0) \in L'$ belong to the same leaf L' of the foliation and, by Proposition 5.0.5, the map f is constant on the leaf L'. Thus, we arrive at the contradiction

$$f(L') = f(x^{-1}(\delta_1, 0\ldots,0)) \neq f(x^{-1}(\delta_2, 0\ldots,0)) = f(L').$$

So, if $L' \cap V' \neq \varnothing$, the intersection must be connected. $\square$

Lemma 5.0.9. *For each point $p \in \mathcal{U}$, there exists a foliated neighborhood $V' \subset \mathcal{U}$ such that $f|_{V'}$ factors through a smooth embedding of an open interval of $\mathbb{R}$ into M:*

$$\begin{tikzcd} V' \arrow[rr, "f|_{V'}"] \arrow[dr] & & M. \\ & (-\delta,\delta) \arrow[ur, hook] & \end{tikzcd}$$

Proof. Let V' be the foliated neighborhood of the point p guaranteed by Proposition 5.0.8. Also, there exists a distributional diffeomorphism,

$$x| : V' \xrightarrow{\cong} (-\delta,\delta) \times (-\varepsilon,\varepsilon)^{n-1}$$

as well as the smooth embedding

$$f \circ c : (-\delta,\delta) \hookrightarrow M$$

defined by $f \circ c(t) = f \circ x^{-1}(0,t)$.

Define a smooth map $\psi : V' \longrightarrow (-\delta,\delta)$ via $\psi(p') := \mathsf{pr}_1 \circ x(p')$ for $p' \in V'$. Consider the point

$$c \circ \psi(p') = x^{-1}(0, \mathsf{pr}_1 \circ x(p')).$$

in V' for a point $p' \in V'$. Note that the first coordinates of $x(p')$ and $(0, \mathsf{pr}_1 \circ x(p'))$ in the product space $(-\varepsilon,\varepsilon)^{n-1} \times (-\delta,\delta)$ agree. Thus, these points are in the same leaf of the standard codimension 1 foliation of $(-\varepsilon,\varepsilon)^{n-1} \times (-\delta,\delta)$ and, since x^{-1} maps leafs to leaf, the images of these points

$$p' = x^{-1}(x(p')) \text{ and } c \circ \psi(p') = x^{-1}(0, \mathsf{pr}_1 \circ x(p'))$$

are in the same leaf in V'. By Proposition 5.0.5, the map f is constant on this leaf

and

$$f(p') = f \circ c \circ \psi(p')$$

Thus, $f|_{V'}$ smoothly factors through the smooth embedding $f \circ c$:

$$\begin{tikzcd} V' \arrow[rr, "f|_{V'}"] \arrow[dr, "\psi"] & & M. \\ & (-\delta, \delta) \arrow[ur, hook, "f\circ c"'] & \end{tikzcd}$$

□

<u>The set of compact leafs $\mathcal{U}'$ is open</u>

We use that the map f factors through an embedding of the real numbers, as per Lemma 5.0.9, to construct a vector field of a neighborhood of a compact leaf. This vector field will brings the chosen leaf to other compact leafs via flowing. This will yield an open set about each compact leaf that is foliated by compact leafs. Thus, the set $\mathcal{U}'$ will be shown to be open.

Proposition 5.0.10. *Let $L \subset \mathcal{U}$ be a compact leaf. There is a vector field X_L defined on an open neighborhood W_L of L such that, for any leaf ℓ that has non-empty intersection with W_L and any pair of point $p, p' \in \ell \cap W_L$, there is an equality of points*

$$f \circ \varphi_t^{X_L}(p) = f \circ \varphi_t^{X_L}(p')$$

for all values t such that the flows of p and p' by the vector field X_L are defined.

Proof. For each point in the compact leaf L, by Lemma 5.0.9, there exists a neighborhood such that f restricted to the neighborhood factors through a smooth embedding of $\mathbb{R}$ into M. Since L is compact, L can be covered by finitely many such

neighborhoods: $V_1, V_2, \ldots, V_k$. Note that each of these neighborhoods is centered at a point in L, and thus $L \subset \cup_{i=1}^k V_i$.

For each i, the image $f(V_i) \subset M$ is an embedded copy of the real numbers. Consider the intersection of these embeddings. Since $L \cap V_i \neq \varnothing$ for each i and $f|_L$ is constant, $f(L) \in \cap_{i=1}^k V_i$. Also, this intersection is an open subset of $f(V_i)$ for each i. Thus, since $f|_{V_i}$ is continuous for each i, the set

$$(f|_{V_i})^{-1}\left(\bigcap_{i=1}^k f(V_i)\right) \subset V_i \subset \mathcal{U}$$

is open in $\mathcal{U}$. Then, the union of these preimages is open in $\mathcal{U}$. Also, the leaf L is contained in this open set:

$$L \subset \bigcup_{i=1}^k (f|_{V_i})^{-1}\left(\bigcap_{i=1}^k f(V_i)\right).$$

Since L is connected, it resides in a single connected component W_L of this union.

The map f restricted to the open set W_L maps into $f(V_1)$, an embedded copy of $\mathbb{R}$ in the manifold M. Let $\phi : f(V_1) \xrightarrow{\cong} \mathbb{R}$ be a diffeomorphism between the embedded space $f(V_1)$ and the real numbers. Then,

$$\phi \circ f|_{W_L} : W_L \longrightarrow \mathbb{R}$$

is a smooth map into $\mathbb{R}$. Since $W_L \subset \mathbb{S}^n$ has a Riemannian metric, the gradient vector field

$$X_L = \nabla(\phi \circ f|_{W_L})$$

is defined.

Let ℓ be a compact leaf that has non-empty intersection with the open set W_L.

Let $p, p' \in \ell \cap W_L$ be two points in the leaf ℓ. By Proposition 5.0.5, the points $f(p)$ and $f(p')$ in $f(W_L) \subset M$ agree, as do the values $\phi \circ f(p)$ and $\phi \circ f(p')$ in the real numbers. Thus, the points p and p' are in the same level set with respect to the map $\phi \circ f$.

Since gradient flow brings level sets to level sets, for any time t such that the flows of p and p' by the vector field X_L are both defined, $\phi \circ f$ evaluated on the the result of the flowing for each point agrees:

$$\phi \circ f(\varphi_t^{X_L}(p)) = \phi \circ f(\varphi_t^{X_L}(p')).$$

As the map ϕ is a diffeomorphism, the desired equality is verified.

□

Proposition 5.0.11. *For each compact leaf $L \subset \mathcal{U}$, there is an open set V_L containing L and a positive value $\varepsilon > 0$ such that flowing L by the vector field X_L yields the open set V_L,*

$$\varphi^{X_L} : (-\varepsilon, \varepsilon) \times L \xrightarrow{\cong} V_L.$$

Furthermore, for each time $t \in (-\varepsilon, \varepsilon)$, flowing L by X_L for time t yields another leaf of the foliation contained in V_L. Also, for any leaf ℓ that has non-empty intersection with V_L, the leaf ℓ is compact and contained within the set V_L. Thus, the flow map φ^{X_L} is a distributional diffeomorphism,

$$\varphi^{X_L} : ((-\varepsilon, \varepsilon) \times L, (-\varepsilon, \varepsilon) \times TL) \xrightarrow{\cong} (V_L, \ker(Df)).$$

Proof. Let $L \subset \mathcal{U}$ be a compact leaf. By Proposition 5.0.10, there exists an open set W_L containing L and a vector field $X_L \in \mathfrak{X}(W_L)$ such that X_L acts like gradient flow.

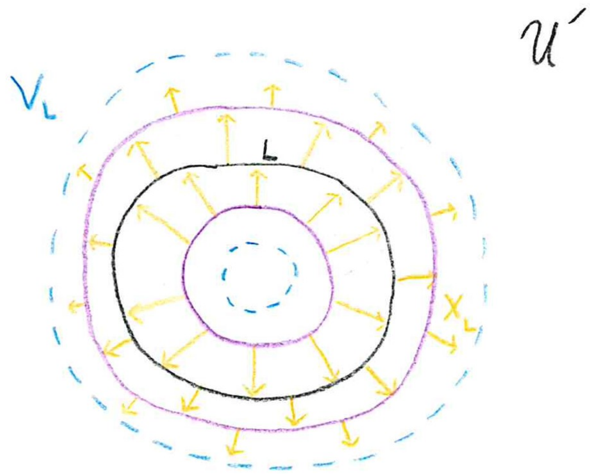

Figure 5.3: The neighborhood V_L of the compact leaf L.

Consider flowing the compact leaf L by the vector field X_L. Since L is compact, there exists a positive value $\varepsilon > 0$ such that any point in L has flow defined for all time between $-\varepsilon$ and ε:

$$\varphi^{X_L} : (-\varepsilon, \varepsilon) \times L \xrightarrow{\cong} V_L,$$

where V_L is defined to be the image of this flow. The map φ^{X_L} is indeed a diffeomorphism as it is defined as the flow of a vector field.

Let t be any value such that $|t| < \varepsilon$. Then, the image of the leaf L via flowing by X_L for time t is diffeomorphic to L by properties of flows; $\varphi_t^{X_L}(L) \cong L$. In particular, the subset $\varphi_t^{X_L}(L)$ of V_L is connected and compact since the leaf L is connected and compact.

Also, via the construction of X_L, for any points $p, p' \in L$, the map f evaluates to the same point in M after flowing:

$$f \circ \varphi_t^{X_L}(p) = f \circ \varphi_t^{X_L}(p').$$

Thus, the image of the leaf L under flowing by X_L for time t is connected and contained in a leaf L' of the foliation. Since L and L' have the same dimension and $\varphi_t^{X_L}$ embeds L into L', it follows that there is an equality of leaves, $\varphi_t^{X_L}(L) = L'$. Therefore, the flow φ^{X_L} brings the leaf L to other compact leafs of the foliation.

Furthermore, as will be shown, any leaf that has non-empty intersection with the set V_L can be recovered in this way. Let ℓ be any compact leaf that has non-empty intersection with the open set V_L. Let $p' \in \ell \cap V_L$ be a point witnessing the non-empty intersection. Then, there exists point $p \in L$ and time $t \in (-\varepsilon, \varepsilon)$ such that flowing p by vector field X_L for time t yields the point $p' = \varphi_t^{X_L}(p)$.

But, flowing L by time t yields another compact leaf of the foliation: $\varphi_t^{X_L}(L) \subset \mathcal{U}$. Since leafs form a partition of the foliated space and the point p' is in both leaves ℓ and $\varphi_t^{X_L}(L)$, there is an equality of leaves: $\varphi_t^{X_L}(L) = \ell$. Thus, every leaf that has non-empty intersection with V_L is contained within V_L. So, V_L is foliated by compact leaves.

Finally, we verify that the diffeomorphism φ^{X_L} is indeed distributional, that is,

$$\begin{tikzcd}
(-\varepsilon,\varepsilon) \times TL \arrow[r, dashed] \arrow[d, hook] & \ker(Df) \arrow[d, hook] \\
T((-\varepsilon,\varepsilon) \times L) \arrow[r, "D\varphi^{X_L}"] \arrow[d] & TV_L \arrow[d] \\
(-\varepsilon,\varepsilon) \times L \arrow[r, "\varphi^{X_L}"] & V_L.
\end{tikzcd}$$

Let (t, p) be a point in $(-\varepsilon, \varepsilon) \times L$ and take a tangent vector

$$Y_p \in T_p L \subset T_{(t,p)}((-\varepsilon,\varepsilon) \times L).$$

There exists a smooth path

$$c : (-\delta, \delta) \longrightarrow L \overset{(t, \mathbb{1}_L)}{\hookrightarrow} (-\varepsilon, \varepsilon) \times L$$

such that $c(0) = (t, p)$ and $c'(0) = Y_p$.

Consider the path

$$\varphi^{X_L} \circ c : (-\delta, \delta) \longrightarrow V_L,$$

and its image under the map f. For all times $s \in (-\delta, \delta)$, since the time t is fixed for $c(s)$, the path $f \circ \varphi^{X_L} \circ c(s)$ is constant. So, the derivative of this map is zero:

$$(f \circ \varphi^{X_L} \circ c)'(s) = f' \circ (\varphi^{X_L} \circ c)'(s) = 0.$$

Thus, by chain rule, the tangent vector $(\varphi^{X_L} \circ c)'(0) \in T_{\varphi_t^{X_L}(p)} V_L$ is mapped via the linear map $D_{\varphi_t^{X_L}(p)} f$ to zero. Therefore, the linear map $D_{(t,p)} \varphi^{X_L}$ takes image in the kernel of Df,

$$D_{(t,p)} \varphi^{X_L}(Y_p) = (\varphi^{X_L} \circ c)'(0) \in \ker(D_{\varphi_t^{X_L}(p)} f),$$

and the diffeomorphism φ^{X_L} is a distributional map.

□

Corollary 5.0.12. *The collection of compact leafs $\mathcal{U}'$ is an open set in $\mathbb{S}^n$.*

Proof. Let $p \in \mathcal{U}'$ be a point contained in the collection of compact leafs of the foliation of $\mathcal{U}$. Thus, there exists compact leaf L which contains the point p. By Proposition 5.0.11, there exists an open set V_L which contains L and is foliated by compact leafs. Thus, the set V_L is an open neighborhood of p that is contained in the collection of compact leafs $\mathcal{U}'$.

□

Observation 5.0.13. Since the collection of compact leafs $\mathcal{U}'$ is open, it is a manifold and $\ker(Df)$ on $\mathcal{U}'$ is an integrable distribution. Thus, it is correct to say that the open set $\mathcal{U}'$ is foliated by compact leafs.

The leaf space of $\mathcal{U}'$

We now look to see how the compact leafs of the open set $\mathcal{U}'$ organize. We will show that the foliation on $\mathcal{U}'$ is quite rigid in that, given a connected component of the open set $\mathcal{U}'$, the foliation is determined by any one of the leafs of the given component. More specifically, if L is a compact leaf in the given connected component, then the foliation is equivalent to the foliation of $\mathbb{R} \times L$ by leafs $\{t\} \times L$ for each value $t \in \mathbb{R}$.

In order to make this precise, we will define the *leaf space* of the open set $\mathcal{U}'$ where elements of the leaf space are the compact leafs of $\mathcal{U}'$. We will then show that the leaf space has some nice properties, in particular, being a 1-manifold that the set $\mathcal{U}'$ fibers over. Using properties of fiber bundles, it will be shown that the leaf space is diffeomorphic to a countable collection of copies of $\mathbb{R}$ and that each component of the set $\mathcal{U}'$ has a foliation determined by one of its leafs.

Definition of the leaf space

Definition 5.0.14. Define an equivalence relation $\sim$ on $\mathcal{U}'$ by declaring points $p, p' \in \mathcal{U}'$ to be equivalent $p \sim p'$ if there exists a compact leaf $L \subset \mathcal{U}'$ such that p and p' are both in L.

This is indeed an equivalence relation as the foliation of $\mathcal{U}'$ yields a partition of the open set.

Denote by $q(\mathcal{U}')$ the quotient space of the open set $\mathcal{U}'$ with respect to the equivalence relation $\sim$. The quotient space $q(\mathcal{U}')$ is referred to as the *leaf space* of $\mathcal{U}'$

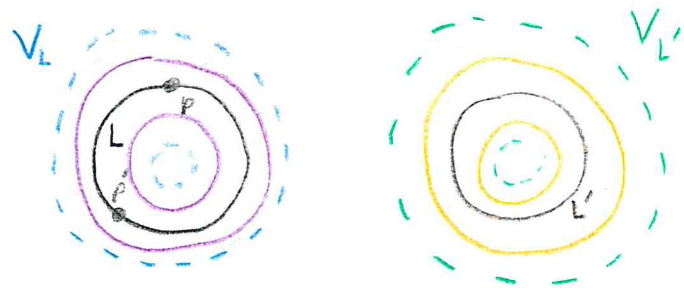

Figure 5.4: L and L' represent elements of $q(\mathcal{U}')$.

as its elements are the leafs of the foliation of $\mathcal{U}'$. Let

$$q : \mathcal{U}' \longrightarrow q(\mathcal{U}')$$

be the continuous quotient map.

Notation 5.0.15. Elements of the leaf space $q(\mathcal{U}')$ are precisely the compact leafs of the foliation of the open set $\mathcal{U}'$. Thus, if a given equivalence class in $q(\mathcal{U}')$ represents a leaf $L \subset \mathcal{U}'$, we will identify the equivalence class by $[L]$. This will also help to differentiate when we are discussing a leaf L as a compact leaf in the space $\mathcal{U}'$ versus discussing the equivalence class $[L]$ in the leaf space $q(\mathcal{U}')$.

Also, note that with this convention that, for any leaf L, that $q(L) = [L]$ and, if p is an element of the leaf L, that $q(p) = [L]$.

The leaf space is a 1-manifold

Lemma 5.0.16. *The quotient map $q : \mathcal{U}' \longrightarrow q(\mathcal{U}')$ is an open map and $q(\mathcal{U}')$ is second-countable.*

Proof. First, we argue that q is an open map.

Let $U \subset \mathcal{U}'$ be an open subset of the compactly-foliated space $\mathcal{U}'$. The image $q(U)$ is the collection of leafs in $\mathcal{U}'$ that intersect U. Thus, $q^{-1}(q(U))$ is the union of all leafs in $\mathcal{U}'$ that have non-empty intersection with the open set U. We will argue that this union of leafs is an open set.

Let $p \in q^{-1}(q(U))$. Since $\mathcal{U}'$ is foliated by compact leafs, there exists a leaf L containing p. Since $p \in q^{-1}(q(U))$, this leaf has non-empty intersection with U.

Let X be the vector field guaranteed by Lemma 5.0.10. As noted in Corollay 5.0.11, there exists some real number $\varepsilon > 0$ such that flowing L by X for any time $|t| < \varepsilon$ yields another leaf of the foliation.

Restrict the vector field X to the open set U. Flowing by $X|_U$ yields a partially defined map,

$$\varphi^{X|_U} : \mathbb{R} \times U \dashrightarrow U.$$

Take a point $p' \in L \cap U$. Flowing the the point p' by $X|_U$ is defined on some interval

$$\varphi_{-}^{X|_U}(p') : (-\delta, \delta) \to U$$

for some $\delta > 0$.

Take $\varepsilon' := \min(\varepsilon, \delta)$. Then the image of the flow

$$\varphi^X|_{(-\varepsilon', \varepsilon') \times L} : (-\varepsilon', \varepsilon') \times L \longrightarrow \mathcal{U}'$$

is an open subset of $\mathcal{U}'$ that contains the leaf L, and thus is a neighborhood of the point p. Also, for each $|t| < \varepsilon'$, the leaf

$$\varphi_t^X(L) \subset \mathcal{U}'$$

has non-empty intersection with U. Indeed, since $|t| < \varepsilon' \leq \delta$ and $p' \in L$,

$$\varphi_t^X(p') = \varphi_t^{X|_U}(p') \in U.$$

Thus, every leaf of $\mathrm{Im}(\varphi^X|_{(-\varepsilon',\varepsilon')\times L})$ has non-empty intersection with U and is therefore contained in $q^{-1}(q(U))$. By arbitrary choice of p, the set $q^{-1}(q(U))$ is open and q is an open map.

Finally, since q is an open quotient map and $\mathcal{U}'$ is second-countable, the leaf space $q(\mathcal{U}')$ is second-countable as well. □

Lemma 5.0.17. *The leaf space $q(\mathcal{U}')$ is Hausdorff.*

Proof. Let $q(L)$ and $q(L')$ be distinct elements of the leaf space $q(\mathcal{U}')$. Each of these elements has their corresponding compact leaf in $\mathcal{U}'$, called L and L', respectively.

Since $\mathcal{U}'$ is a manifold, it is a normal space and there exists disjoint open sets $U, U' \subset \mathcal{U}'$ such that $L \subset U$ and $L' \subset U'$.

Construct a vector field X on a subset of U that contains L that respects the foliation, as per Proposition 5.0.10. Flowing L by X yields an open subset U_L of U that is foliated by copies of L.

Likewise construct a vector field X' on a subset of U' that contains L' that respects the foliation. Flowing L' by X' yields an open subset $U'_{L'}$ of U' that is foliated by copies of L'. See Figure 5.4.

Since U_L and $U'_{L'}$ are contained in mutually disjoint open sets U and U', these constructed neighborhoods of L and L' do not intersect. Moreover, for both U_L and $U'_{L'}$, if a leaf has non-empty intersection with the open set, that leaf is contained in the open set. So, there exists no leaf that has non-empty intersection with both U_L and $U'_{L'}$.

Therefore, the images of these open sets $q(U_L)$ and $q(U'_{L'})$ are mutually disjoint, open neighborhoods of $q(L)$ and $q(L')$ in the leaf space $q(\mathcal{U}')$.

□

Proposition 5.0.18. *The leaf space $q(\mathcal{U}')$ can be endowed with a smooth structure, making it a 1-dimensional manifold, such that the quotient map q is smooth.*

Proof. It has been shown in Lemma 5.0.16 and Lemma 5.0.17 that $q(\mathcal{U}')$ is a second countable, Hausdorff space. It remains to construct a smooth atlas on $q(\mathcal{U}')$.

By Lemma 5.0.16, for each compact leaf $L \subset \mathcal{U}'$, the image of the open set V_L guaranteed by Corollary 5.0.11 under the map q is open in $q(\mathcal{U}')$. The idea is to show that $q(V_L)$ for each compact leaf $L \subset \mathcal{U}'$ is a coordinate neighborhood with a yet to be defined homomorphism ϕ_L.

Fix a compact leaf $L \subset \mathcal{U}'$ and consider the open set $q(V_L) \subset q(\mathcal{U}')$. Recall that there exists a vector field X_L defined on a neighborhood of L such that the associated flow yields a distributional diffeomorphism:

$$\varphi^{X_L} : ((-\varepsilon, \varepsilon) \times L, (-\varepsilon, \varepsilon) \times TL) \xrightarrow{\cong} (V_L, \Delta)$$

for some $\varepsilon > 0$. In particular, for any leaf $\ell \subset V_L$, the smooth map

$$\mathsf{pr}_1 \circ (\varphi^{X_L})^{-1} : V_L \longrightarrow (-\varepsilon, \varepsilon)$$

is constant on ℓ.

Define a real-valued map ϕ_L on $q(V_L)$:

$$\phi_L : q(V_L) \longrightarrow (-\varepsilon, \varepsilon)$$

$$[\ell] \longmapsto \mathsf{pr}_1 \circ (\varphi^{X_L})^{-1}(\ell)$$

The map ϕ_L is well-defined since each element of $q(V_L)$ corresponds to exactly one leaf of the foliation of V_L and the map $\mathsf{pr}_1 \circ (\varphi^{X_L})^{-1}$ is constant on each leafs of the foliation.

We will show that ϕ_L is continuous. Let $(\delta, \delta') \subset (-\varepsilon, \varepsilon)$ be an open subset of the target of the map ϕ_L. Then, the set $(\delta, \delta') \times L \subset (-\varepsilon, \varepsilon) \times L$ is open and its image under the diffeomorphism φ^{X_L} is an open subset of V_L. In particular, since φ^{X_L} is a distributional diffeomorphism and $(\delta, \delta') \times L$ is foliated by compact leafs, where the leafs are given by $\{t\} \times L$ for $t \in (\delta, \delta')$, the image

$$\varphi^{X_L}((\delta, \delta') \times L) \subset V_L$$

is an open set that is foliated by compact leafs. Since q is an open map, the set

$$q \circ \varphi^{X_L}((\delta, \delta') \times L) \subset q(\mathcal{U}')$$

is open in the leaf space. In fact, this set is the preimage of the open set (δ, δ') under the map ϕ_L. Thus, the set $(\phi_L)^{-1}(\delta, \delta')$ is open in $q(\mathcal{U}')$ and the map ϕ_L is continuous.

Now, we define a continuous map ϕ_L^{-1} and show, as the notation indicates, that it is the inverse of the continuous map ϕ_L. Fix a point $p \in L$. Define a map from an open interval to $q(V_L)$,

$$\phi_L^{-1} := q \circ \varphi^{X}(p, -) : (-\varepsilon, \varepsilon) \longrightarrow q(V_L).$$

The map ϕ_L^{-1} is continuous as it is the composition of continuous maps,

$$\begin{tikzcd}
(-\varepsilon,\varepsilon) \arrow[r, hook, "\mathbb{1}_{(-\varepsilon,\varepsilon)}\times\{p\}"] \arrow[drrr, "\phi_L^{-1}"'] & (-\varepsilon,\varepsilon)\times L \arrow[r, "\varphi^X", "\cong"'] & V_L \arrow[r, hook] \arrow[d, "q|_{V_L}"] & \mathcal{U}' \arrow[d, "q"] \\
 & & q(V_L) \arrow[r, hook] & q(\mathcal{U}') .
\end{tikzcd}$$

Now, we verify that these maps are inverse of each other. Let $t \in (-\varepsilon,\varepsilon)$ and consider

$$\phi_L \circ \phi_L^{-1}(t) = \phi_L \circ q \circ \varphi^X(p,t)$$

The equivalence class $q \circ \varphi^X(p,t)$ is a single leaf ℓ in $\mathcal{U}'$ which contains the point $\varphi^X(p,t)$. Thus, ℓ is the image of $\{t\} \times L$ under the map φ^{X_L}. Therefore, ϕ_L returns the value t:

$$\phi_L \circ q \circ \varphi^X(p,t) = t.$$

Consider the elements $[\ell]$ and $\phi_L^{-1} \circ \phi_L([\ell])$ in $q(V_L)$. By definition, the value $\phi_L([\ell])$ is such that the image of $\{\phi_L([\ell])\} \times L$ under the map φ^{X_L} is the leaf ℓ associated to the equivalence class. Thus, $\varphi^{X_L}(\phi_L([\ell]),p)$ is an element of ℓ, where p was the specified point in L. So, the elements $\phi_L^{-1} \circ \phi_L([\ell])$ and $[\ell]$ of $q(V_L)$ agree. Therefore, ϕ_L and ϕ_L^{-1} are inverses of each other and each map is a homeomorphism.

Consider the collection

$$\mathcal{A} := \{(\phi_L, q(V_L)) \mid q(L) \in q(\mathcal{U}')\}.$$

The collection of open sets $q(V_L)$ cover $q(\mathcal{U}')$. In order to verify that $\mathcal{A}$ is a smooth atlas, we need to verify that all transition maps are smooth.

Let L and L' be compact leafs in the foliated space $\mathcal{U}'$ and take the associated charts, $(\phi_L, q(V_L))$ and $(\phi_{L'}, q(V_{L'}))$. Consider the continuous map

$$\theta_{L,L'} := \phi_L \circ \phi_{L'}^{-1}|_{\phi_{L'}(q(V_L)\cap q(V_{L'}))} : \phi_{L'}(q(V_L) \cap q(V_{L'})) \longrightarrow (-\varepsilon,\varepsilon).$$

Let t be any value in the domain of the map $\theta_{L,L'}$. Denote by p' the element of L' used to define the map $\phi_{L'}^{-1}$. Then, the image of t under $\phi_{L'}^{-1}$ in $q(V_L) \cap q(V_{L'})$ is the leaf ℓ of $\mathcal{U}'$ that contains the element $\varphi_t^{X_{L'}}(p')$.

Thus, the image of the value t under the map $\theta_{L,L'}$ is the value $\mathsf{pr}_1 \circ (\varphi^{X_L})^{-1}(\ell)$. But, since the map $\mathsf{pr}_1 \circ (\varphi^{X_L})^{-1}$ is constant on leafs, we have an equality of values,

$$\theta_{L,L'}(t) = \mathsf{pr}_1 \circ (\varphi^{X_L})^{-1}(\ell) = \mathsf{pr}_1 \circ (\varphi^{X_L})^{-1} \circ \varphi_t^{X_{L'}}(p').$$

Therefore, the map $\theta_{L,L'}$ can be written as a composition of smooth maps,

$$\begin{tikzcd} \phi_{L'}(q(V_L) \cap q(V_{L'})) \arrow[r, hook, "\mathbb{1}\times\{p'\}"] \arrow[rrrr, bend right, "\theta_{L,L'}"'] & (-\varepsilon', \varepsilon') \times L' \arrow[r, "\varphi^{X_{L'}}"] & V_L \cap V_{L'} \arrow[r, "(\varphi^{X_L})^{-1}"] & (-\varepsilon, \varepsilon) \times L \arrow[r, "\mathsf{pr}_1"] & (-\varepsilon, \varepsilon), \end{tikzcd}$$

and is thus smooth. Therefore, $\mathcal{A}$ is a smooth atlas and $q(\mathcal{U}')$ is a smooth 1-manifold. The quotient map q is shown to be smooth in Proposition 5.0.19. $\square$

<u>$q : \mathcal{U}' \longrightarrow q(\mathcal{U}')$ is a smooth fiber bundle</u>

Proposition 5.0.19. *The map $q : \mathcal{U}' \longrightarrow q(\mathcal{U}')$ is a smooth submersion.*

Proof. This follows nearly immediately from $q(\mathcal{U}')$ being a 1-manifold. The map q effectively acts like projection onto the "time" coordinate with respect to the flowing argument outlined in Corollary 5.0.11.

Let $p \in \mathcal{U}'$. There exists a leaf of the foliation $L \subset \mathcal{U}'$ such that $p \in L$. Take a locally trivial foliation neighborhood $L \subset V_L$ guaranteed via Corollary 5.0.11; that is there exists open set $V_L \subset \mathcal{U}'$ containing L and positive value $\varepsilon > 0$ such that the

distributional diffeomorphism

$$\varphi^{X_L} : (-\varepsilon, \varepsilon) \times L \xrightarrow{\cong} V_L$$

exists.

As was argued in Proposition 5.0.18, there is a coordinate neighborhood $(\phi_L, q(V_L))$ of the equivalence class $q(p)$ in the leaf space with associated diffeomorphism

$$\begin{aligned} \phi_L : q(V_L) &\longrightarrow (-\varepsilon, \varepsilon) \\ [\ell] &\longmapsto \mathsf{pr}_1 \circ (\varphi^{X_L})^{-1}(\ell). \end{aligned}$$

We will now use the map φ^{X_L} to construct a coordinate neighborhood of $p \in \mathcal{U}'$. Let (x, U) be a coordinate neighborhood of p in the smooth manifold L. So, there exists positive value $\delta > 0$ such that

$$x : U \longrightarrow (-\delta, \delta)^{n-1}$$

is a diffeomorphism.

Take the coordinate neighborhood $(x_L, \varphi^{X_L}((-\varepsilon, \varepsilon) \times U))$ about the point p where the map x_L is defined by the composition

$$x_L : \varphi^{X_L}((-\varepsilon, \varepsilon) \times U) \xrightarrow{\varphi^{X_L}|^{-1}_{(-\varepsilon,\varepsilon)\times U}} (-\varepsilon, \varepsilon) \times U \xrightarrow{(\mathbb{1}_{(-\varepsilon,\varepsilon)}, x)} (-\varepsilon, \varepsilon) \times (-\delta, \delta)^{n-1}.$$

Now, it will be argued that the map

$$\phi_L \circ q \circ x_L^{-1} : (-\varepsilon, \varepsilon) \times (-\delta, \delta)^{n-1} \longrightarrow (-\varepsilon, \varepsilon)$$

is smooth and, in fact, a submersion.

Let $(t_1, t_2, \ldots, t_n) \in (-\varepsilon, \varepsilon) \times (-\delta, \delta)^{n-1}$. Evaluating the map $q \circ x_L^{-1}$ at this point yields the equivalence class

$$q\left(\varphi_{t_1}^{X_L} \circ x^{-1}(t_2, \ldots, t_n)\right) \in q(\mathcal{U}').$$

Since $\varphi_{t_1}^{X_L} \circ x^{-1}(t_2, \ldots, t_n)$ is contained in the leaf $\varphi_{t_1}^{X_L}(L)$ of the foliation of V_L, this equivalence class is the leaf $\varphi_{t_1}^{X_L}(L)$. Thus, since the map $\mathsf{pr}_1 \circ (\varphi^{X_L})^{-1}$ is constant on leafs of the foliation, we arrive at the equality of values,

$$\phi_L \circ q \circ x_L^{-1}(t_1, \ldots, t_n) = \mathsf{pr}_1 \circ (\varphi^{X_L})^{-1}\left(q\left(\varphi_{t_1}^{X_L} \circ x^{-1}(t_2, \ldots, t_n)\right)\right) = t_1.$$

Therefore, the map $\phi_L \circ q \circ x_L^{-1}$ is the projection map onto the first coordinate of its domain. So, it is smooth and a submersion. Thus, the map q is a smooth submersion. $\square$

Proposition 5.0.20. *The map $q : \mathcal{U}' \longrightarrow q(\mathcal{U}')$ is proper.*

Proof. Let $B \subset q(\mathcal{U}')$ be a compact subset of the leaf space. We will show that $q^{-1}(B)$ is (sequentially) compact.

Let (p_n) be a sequence of points in $q^{-1}(B)$. Since the image of the sequence is contained in the compact set B, there is a subsequence (p_{n_k}) such that its image in the leaf space converges to some element $[L] \in B$:

$$q(p_{n_k}) \xrightarrow{n_k \to \infty} [L].$$

Take the neighborhood V_L of the compact leaf L guaranteed by Corollary 5.0.11. The open set V_L is diffeomorphic to $(-\varepsilon, \varepsilon) \times L$ for some $\varepsilon > 0$,

$$\varphi^{X_L} : (-\varepsilon, \varepsilon) \times L \xrightarrow{\cong} V_L.$$

Consider the following sets defined as the image of restrictions of this diffeomorphism:

$$V'_L = \varphi^{X_L}((-\varepsilon/2, \varepsilon/2) \times L) \text{ and } \overline{V'_L} = \varphi^{X_L}([-\varepsilon/2, \varepsilon/2] \times L).$$

The open set $q(V'_L)$ in the leaf space is an open neighborhood of $[L]$. Thus, there exists an $N \in \mathbb{N}$ such that, for all $n_k \geq N$, the sequence $q(p_{n_k})$ is in $q(V'_L)$. Thus, the associated subsequence (p_{n_k}) is in V'_L for all $N_k \geq N$. In fact, since the set $\overline{V'_L}$ is compact, it can be assumed that the sequence (p_{n_k}) converges to a point p in $\overline{V'_L}$.

Finally, since q is continuous, the image of this sequence converges in the leaf space:

$$q(p_{n_k}) \overset{n_k \to \infty}{\longrightarrow} q(p).$$

As the leaf space is Hausdorff, this sequence converges to a unique element of the leaf space and there is an equality of equivalence classes: $q(p) = [L]$. Thus, p is an element of L. Since L is a subset of the preimage $q^{-1}(B)$, the sequence (p_n) in $q^{-1}(B)$ has a subsequence that converges in $q^{-1}(B)$. So, the preimage by the map q of the compact set B is compact.

□

Corollary 5.0.21. $q : \mathcal{U}' \longrightarrow q(\mathcal{U}')$ *is a smooth fiber bundle.*

Proof. By Propositions 5.0.19 and 5.0.20, q is a smooth and proper submersion. As its target is the quotient space associated to the map, q is surjective. Ehresmann's fibration theorem states that any smooth, proper, surjective, submersion is a smooth fiber bundle. Thus, q is a smooth fiber bundle. □

<u>$\mathcal{U}'$ is not the entirety of $\mathbb{S}^n$</u>

We will use the fiber bundle that the leaf space provides to make conclusions about the global structure of the open set $\mathcal{U}'$ in $\mathbb{S}^n$. First, we need to make some

observations about the topology of $\mathcal{U}'$. Namely, that the set $\mathcal{U}'$ cannot be all of $\mathbb{S}^n$. This will be shown via contradiction. First, a lemma about fiber bundles over the 1-sphere, namely that such fiber bundles cannot have simply connected total space, will be shown. It will then follow that the set $\mathcal{U}'$ cannot be the simply connected space $\mathbb{S}^n$.

Proposition 5.0.22. *Let $\pi : \mathcal{E} \longrightarrow \mathbb{S}^1$ be a fiber bundle where $\mathcal{E}$ is a connected, compact manifold. Then $\mathcal{E}$ is not simply connected.*

Proof. Let $\tilde{\mathcal{E}}$ be the universal cover of $\mathcal{E}$. Then, as will be shown, there is a map induced by the fiber bundle $\pi : \mathcal{E} \longrightarrow \mathbb{S}^1$ between their universal covers, $\tilde{\pi} : \tilde{\mathcal{E}} \longrightarrow \mathbb{R}$.

First, pullback the universal cover of $\mathbb{S}^1$ along the fiber bundle $\pi : \mathcal{E} \longrightarrow \mathbb{S}^1$:

$$\begin{tikzcd}
\mathcal{E} \times_{\mathbb{S}^1} \mathbb{R} \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{R} \arrow[d, "\exp"] \\
\mathcal{E} \arrow[r, "\pi"] & \mathbb{S}^1.
\end{tikzcd}$$

The smooth map $b : \mathcal{E} \times_{\mathbb{S}^1} \mathbb{R} \longrightarrow \mathcal{E}$ is a covering space of $\mathcal{E}$ since it is the pullback of the covering map exp by the smooth map π. Similarly, the smooth map $a : \mathcal{E} \times_{\mathbb{S}^1} \mathbb{R} \longrightarrow \mathbb{R}$ is a fiber bundle over $\mathbb{R}$ since it is the pullback of the fiber bundle map π by the smooth map exp.

As $\mathcal{E} \times_{\mathbb{S}^1} \mathbb{R}$ is a covering space of the space $\mathcal{E}$, by the universal property of universal covering spaces, the space $\mathcal{E} \times_{\mathbb{S}^1} \mathbb{R}$ receives a covering map c from $\tilde{\mathcal{E}}$:

$$\begin{tikzcd}
\tilde{\mathcal{E}} \arrow[dr, dashed, "\exists!", "c"'] \arrow[drr, dotted, bend left, "\tilde{\pi}"] \arrow[ddr, bend right] & & \\
& \mathcal{E} \times_{\mathbb{S}^1} \mathbb{R} \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{R} \arrow[d, "\exp"] \\
& \mathcal{E} \arrow[r, "\pi"] & \mathbb{S}^1.
\end{tikzcd}$$

So, the smooth map $\tilde{\pi} = a \circ c$ is a fiber bundle over $\mathbb{R}$ as it is the composition of fiber bundles.

As $\tilde{\pi} : \tilde{\mathcal{E}} \longrightarrow \mathbb{R}$ is a fiber bundle over the contractible space $\mathbb{R}$, it must be a trivial fiber bundle:

$$\tilde{\mathcal{E}} \cong \mathbb{R} \times F$$

for some smooth $(dim(\tilde{\mathcal{E}}) - 1)$-manifold F. Consequently, the higher homology groups vanish. In particular,

$$H_{dim(\tilde{\mathcal{E}})}(\tilde{\mathcal{E}}) = 0.$$

Since $\tilde{\mathcal{E}}$ is the universal cover of the space $\mathcal{E}$, the dimensions of these spaces agrees and the top homology group can be expressed as $H_{dim(\mathcal{E})}(\tilde{\mathcal{E}}) = H_{dim(\tilde{\mathcal{E}})}(\tilde{\mathcal{E}}) = 0$.

If $\mathcal{E}$ was simply connected, the canonical map yields an isomorphism $\tilde{\mathcal{E}} \cong \mathcal{E}$. Since simply connected implies orientable, the top homology group is non-trivial:

$$H_{dim(\mathcal{E})}\left(\tilde{\mathcal{E}}\right) \cong H_{dim(\mathcal{E})}\left(\mathcal{E}\right) \cong \mathbb{Z}.$$

This contradicts that the top homology group of the universal cover $\tilde{\mathcal{E}}$ vanishes. $\square$

Corollary 5.0.23. *The set of compact leafs $\mathcal{U}'$ is not the entirety of $\mathbb{S}^n$.*

Proof. Suppose by way of contradiction that $\mathbb{S}^n$ is foliated by compact leafs, that is $\mathcal{U}' = \mathbb{S}^n$. Then, the total space of the fiber bundle

$$q : \mathcal{U}' \longrightarrow q(\mathcal{U}')$$

is compact and connected. Thus, the leaf space $q(\mathcal{U}')$ is also connected and compact. Since the leaf space $q(\mathcal{U}')$ is a smooth 1-manifold, by classification of 1-manifolds, the leaf space is diffeomorphic to the 1-sphere; $q\left(\mathcal{U}'\right) \cong \mathbb{S}^1$

So, the space of compact leafs $\mathcal{U}'$ fibers over $\mathbb{S}^1$. But, $\mathcal{U}'$ would also be simply connected, contradicting Proposition 5.0.22. □

The foliation on each component of $\mathcal{U}'$ is determined by a leaf

The leaf space of the set $\mathcal{U}'$ will now be used to establish global structure for $\mathcal{U}'$. Results will follow from verifying that all components of the leaf space are noncompact and thus diffeomorphic to $\mathbb{R}$.

Let $\{\mathcal{U}'_\alpha\}_{\alpha\in J}$ be the collection of connected components of the open set $\mathcal{U}'$ for some indexing set J;

$$\mathcal{U}' = \coprod_{\alpha\in J} \mathcal{U}'_\alpha.$$

Since $\mathcal{U}'$ is an open subset of the manifold $\mathbb{S}^n$, the indexing set J is countable.

Proposition 5.0.24. *The leaf space $q(\mathcal{U}')$ is diffeomorphic to a countable disjoint union of copies of $\mathbb{R}$,*

$$q(\mathcal{U}') \cong \coprod_{\alpha\in J} \mathbb{R}.$$

Indeed, each component of $q(\mathcal{U}')$ is diffeomorphic to $\mathbb{R}$.

Proof. If the open set $\mathcal{U}'$ is empty, the result is immediate. Assume then that $\mathcal{U}'$ is a non-empty set. By Corollary 5.0.23, the set $\mathcal{U}'$ is not all of $\mathbb{S}^n$.

Suppose that the set $\mathcal{U}'$ had a compact path component. The component would be closed and open and it would make up all of $\mathbb{S}^n$ or form a separation of $\mathbb{S}^n$. Since $\mathbb{S}^n$ is connected, no such component can exist and the set $\mathcal{U}'$ has no compact components.

Let $A \subset q(\mathcal{U}')$ be a path-component of the leaf space $q(\mathcal{U}')$. We aim to exhibit that the set $q^{-1}(A)$ is a path-component of $\mathcal{U}'$.

Let p and p' be points in the preimage $q^{-1}(A)$. The images of these points lie in

the path-component A. Since A is path-connected, there exists a path

$$\gamma : [0,1] \longrightarrow A$$

such that $\gamma(0) = q(p)$ and $\gamma(1) = q(p')$.

As $q : \mathcal{U}' \longrightarrow q(\mathcal{U}')$ is a fiber bundle, q has the path lifting property. So, there exists a lift of the path γ to a path

$$\tilde{\gamma} : [0,1] \longrightarrow q^{-1}(A)$$

such that $\tilde{\gamma}(0) = p$ and $\tilde{\gamma}(1) \in q^{-1}(q(p'))$.

Since the preimage of a point in $q(\mathcal{U}')$ is a connected leaf in the foliated space $\mathcal{U}'$, the preimage $q^{-1}(q(p'))$ of the equivalence class $q(p')$ is a connected submanifold in $\mathcal{U}'$ which is path-connected. So, there exists a path

$$\beta : [0,1] \longrightarrow q^{-1}(q(p'))$$

such that $\beta(0) = \tilde{\gamma}(1)$ and $\beta(1) = p'$. Concatenating these paths yields a path $\tilde{\gamma} * \beta$ in the set $q^{-1}(A)$ between points p and p'. Since these points were arbitrary, $q^{-1}(A)$ is path-connected.

It still needs to be verified that $q^{-1}(A)$ is an entire path component of $\mathcal{U}'$. Suppose that the point $r \in \mathcal{U}'$ is in the same path component of $q^{-1}(A)$. We will show that the point r is also in the set $q^{-1}(A)$.

Take any point $p \in q^{-1}(A)$. Then, there is a path β in $\mathcal{U}'$ connecting points p and r. Then, the composition $q \circ \beta$ is a path in $q(\mathcal{U}')$ connecting $q(p)$ and $q(r)$. Thus, the equivalence classes $q(p)$ and $q(r)$ lie in the same path component of $q(\mathcal{U}')$. Since the equivalence class $q(p)$ is in the path component A, the equivalence class $q(r)$ is in A

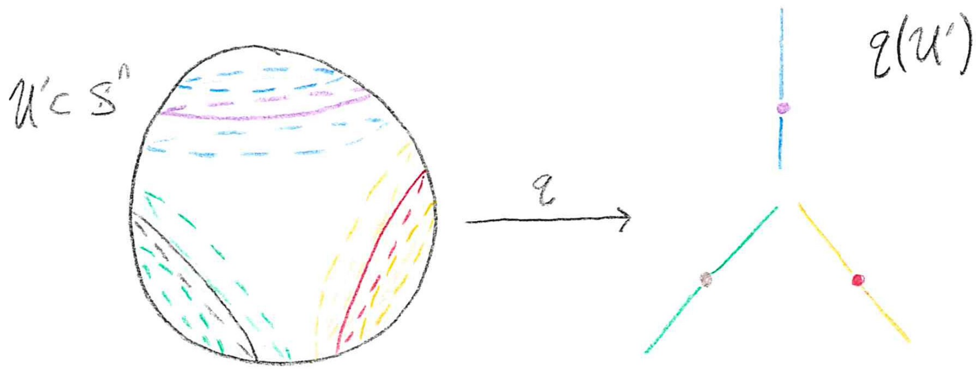

Figure 5.5: Example of a leaf space.

as well. Therefore, r is in the set $q^{-1}(A)$, which represents an entire path component of the set $\mathcal{U}'$.

So, each path component of the leaf space $q(\mathcal{U}')$ determines a path component of the set $\mathcal{U}'$ via preimage of q. Suppose the leaf space $q(\mathcal{U}')$ had a compact path component A. Then, the set $q^{-1}(A)$ would be a path component of the set $\mathcal{U}'$. Since the map q is proper (Proposition 5.0.20), this path component of $\mathcal{U}'$ would be compact, contradicting that the set $\mathcal{U}'$ has no compact path components.

So, each component of the smooth 1-dimensional manifold $q(\mathcal{U}')$ is not compact. Therefore, each component is diffeomorphic to $\mathbb{R}$ by classification of smooth 1-dimensional manifolds.

$\square$

Observation 5.0.25. Along the way in proving Proposition 5.0.24, it is shown that there is a bijection between path components of the space of compact leafs $\mathcal{U}'$ and the leaf space $q(\mathcal{U}')$ given by the map q:

$$\begin{array}{ccc} \pi_0(q(\mathcal{U}')) & \xrightarrow{\cong} & \pi_0(\mathcal{U}') \\ A & \longmapsto & q^{-1}(A). \end{array}$$

Thus, each component of the leaf space $q(\mathcal{U}')$ is determined exactly by a component of $\mathcal{U}'$. As such, we can describe each component of $q(\mathcal{U}')$ as the image $q(\mathcal{U}'_\alpha)$ for some $\alpha \in J$. So, no new notation will be introduced to describe the components of $q(\mathcal{U}')$.

Proposition 5.0.26. *Consider a connected component $\mathcal{U}'_\alpha \subset \mathcal{U}'$ and take a compact leaf contained in the component: $L_\alpha \subset \mathcal{U}'_\alpha$. Then, the component is diffeomorphic to the cylindrical set*

$$\mathcal{U}'_\alpha \cong \mathbb{R} \times L_\alpha.$$

Thus, the set $\mathcal{U}'$ can be described as the disjoint union of cylindrical sets:

$$\mathcal{U}' \cong \coprod_{\alpha \in J} (\mathbb{R} \times L_\alpha).$$

Proof. By Proposition 5.0.24, the component $q(\mathcal{U}'_\alpha)$ of the leaf space is diffeomorphic to $\mathbb{R}$. Thus,

$$q|_{\mathcal{U}'_\alpha} : \mathcal{U}'_\alpha \longrightarrow q(\mathcal{U}'_\alpha)$$

is a fiber bundle over a contractible space and must be a trivial fiber bundle. The fiber over the equivalence class $[L_\alpha]$ in the component $\mathcal{U}'_\alpha$ is the leaf L_α. Therefore, the total space of the fiber bundle $q|_{\mathcal{U}'_\alpha}$ is determined by the leaf L_α:

$$\mathcal{U}'_\alpha \cong q(\mathcal{U}'_\alpha) \times q^{-1}(q(L_\alpha)) \cong \mathbb{R} \times L_\alpha.$$

$\square$

Smooth rank 1 maps of $\mathbb{S}^n$ factor through a tree

We return to constructing the partition of $\mathbb{S}^n$ such that the associated quotient space is a tree and f is constant on the parts of the partition. The desired partition will be made of the compact leafs of the open set $\mathcal{U}'$ and the connected components of the closed set $\mathcal{K}' := \mathbb{S}^n \setminus \mathcal{U}'$. The closed set $\mathcal{K}'$ is the union of the set $\mathcal{K}$ with all non-compact leafs of the foliation of the open set $\mathcal{U}$.

The quasi-foliation of $\mathbb{S}^n$

Definition 5.0.27. Define an equivalence relation $\sim$ on $\mathbb{S}^n$ as follows: points p and p' in $\mathbb{S}^n$ are equivalent, $p \sim p'$ if,

- there exists a compact leaf $L \subset \mathcal{U}'$ containing both p and p' or
- there exists a connected component of $\mathcal{K}'$ containing both p and p'.

This is indeed an equivalence relation as it forms a partition of $\mathbb{S}^n$. The collection of equivalence classes is referred to as a *quasi-foliation* of $\mathbb{S}^n$. Each equivalence class is called a *quasi-leaf*. Let

$$q : \mathbb{S}^n \longrightarrow q(\mathbb{S}^n)$$

be the associated quotient map. The quotient space is called the *quasi-leaf space*.

Example 5.0.28. The quasi-foliation associated to Example 5.0.7 is indicated. The four quasi-leafs associated to $\mathcal{K}'$ are indicated in red. Several of the quasi-leafs associated to the foliation of $\mathcal{U}'$ by compact leafs are indicated in blue.

Observation 5.0.29. The quasifoliation equivalence relation is an extension of the leaf space equivalence relation defined in Definition 5.0.14 to the entire n-sphere.

The map f is indeed constant on each quasi-leaf, as will be shown.

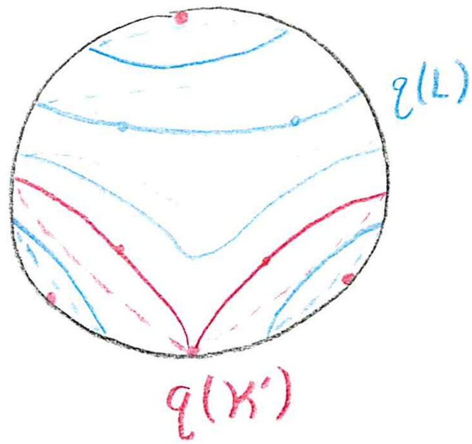

Figure 5.6: Quasi-leaf space associated to Example 5.0.7.

Observation 5.0.30. As was shown in Proposition 5.0.5, the map f is constant on each compact leaf in the set $\mathcal{U}'$. In fact, when f is restricted to any component $\mathcal{U}'_\alpha$ of $\mathcal{U}'$, f factors smoothly through the leaf space $q(\mathcal{U}')$:

$$\begin{tikzcd} \mathcal{U}'_\alpha \arrow[rr, "f|_{\mathcal{U}'_\alpha}"] \arrow[dr, "q"'] & & M. \\ & q(\mathcal{U}'_\alpha) \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

In the case where f is a horizontal map into a contact 3-manifold (M, ξ), the smooth map $\tilde{f}$ will be horizontal.

Lemma 5.0.31. *The map f is constant on components of $\mathcal{K}'$.*

Proof. Let K be a quasi-leaf of $\mathbb{S}^n$ and take a point $p \in K$. Take an open neighborhood $U \subset M$ of the point $f(p)$ with a local parameter $x : U \to \mathbb{R}^3$. Take the connected component of the open subset $f^{-1}(U) \cap K$ that contains p, denoted by V. The subset V is an open neighborhood of p in K. We will show that f is constant on this neighborhood.

Consider the smooth map

$$\mathsf{pr}_i \circ x \circ f|_{f^{-1}(U)} : f^{-1}(U) \longrightarrow \mathbb{R}$$

for $i = 1, 2, 3$. Consider the components of $\mathcal{K} \cap V \subset f^{-1}(U)$. By definition of $\mathcal{K}$, the derivative of f, and thus the derivative of $\mathsf{pr}_i \circ x \circ f|_{f^{-1}(U)}$, vanishes on $\mathcal{K} \cap V$. Then, the subset $\mathcal{K} \cap V$ is contained in the set of critical points of the map $\mathsf{pr}_i \circ x \circ f|_{f^{-1}(U)}$. By Sard's theorem, the image

$$\mathsf{pr}_i \circ x \circ f(\mathcal{K} \cap V) \subset \mathbb{R}$$

has measure zero.

Let L be a connected component of $V \smallsetminus (\mathcal{K} \cap V)$. Since L is a non-compact immersed submanifold contained in V, the closure of L is contained within $\mathcal{K} \cap V$. Let $K' \subset V$ be a connected component of $\mathcal{K} \cap V$ that has non-empty intersection with the closure of L.

Now, f is constant on L (Proposition 5.0.5). Also, f is constant on K' (Proposition 5.0.6). By continuity of f, we have an equality of points $f(L) = f(K')$. Likewise, the ith projections of $f(L)$ and $f(K')$ will agree. Then, for each $L \subset V \smallsetminus (\mathcal{K} \cap V)$, the image of L under the map $\mathsf{pr}_i \circ x \circ f$ is contained in the measure zero set $\mathsf{pr}_i \circ x \circ f(\mathcal{K} \cap V)$. Therefore, we have the set containment

$$\mathsf{pr}_i \circ x \circ f(V) \subset \mathsf{pr}_i \circ x \circ f(\mathcal{K} \cap V).$$

Thus, the image of V under the map $\mathsf{pr}_i \circ x \circ f$ has measure zero in $\mathbb{R}$. Since V is connected, the subset $\mathsf{pr}_i \circ x \circ f(V)$ of $\mathbb{R}$ is connected. Therefore, as the set is connected and measure zero in $\mathbb{R}$, it is a singleton set. So, the map $\mathsf{pr}_i \circ x \circ f|_V$ is

constant. Since this is true for all i, the map $f|_V$ is constant. Thus, $f|_K$ is locally constant. Therefore, since K is connected, the map $f|_K$ is constant.

$\square$

Observation 5.0.32. Thus, the map f is constant on each quasi-leaf of the quasi-foliation of $\mathbb{S}^n$. So the map f factors continuously through the quasi-leaf space $q\left(\mathbb{S}^n\right)$:

$$\begin{tikzcd}
\mathbb{S}^n \arrow[rr, "f"] \arrow[dr, "q"'] & & M. \\
& q(\mathbb{S}^n) \arrow[ur] &
\end{tikzcd}$$

The quasi-leaf space is a tree

The quasi-foliation of $\mathbb{S}^n$ yields a partition in the space such that the map f factors through the quasi-leaf space. We now will show that the quasi-leaf space is a tree.

Each leaf in the set $\mathcal{U}'$ is a codimension 1 compact submanifold of $\mathbb{S}^n$. By the Jordan-Brouwer separation theorem, each compact leaf $L \subset \mathcal{U}'$ separates the space $\mathbb{S}^n$, i.e., the space resulting from removing the leaf $\mathbb{S}^n \setminus L$ has exactly two connected components. This will be enough to show that the equivalence class $[L]$ in the quasi-leaf space is a cut point. It will follow that the quasi-leaf space is a (not necessarily simplicial) tree.

Definition 5.0.33. Let X be a connected topological space and let $A, B, C \subset X$ be subsets of X. The set B *separates* A and C if the complement of B in X has a separation by open sets $W_0, W_1 \subset X$, i.e.,

$$X \setminus B = W_0 \coprod W_1,$$

such that $A \subset W_0$ and $C \subset W_1$.

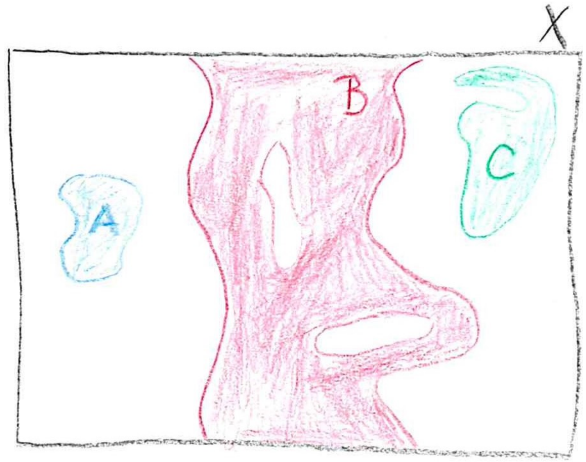

Figure 5.7: B separates A and C in the connected space X.

Lemma 5.0.34. *For each component $\mathcal{U}'_\alpha \subset \mathcal{U}'$, there is an open set $B_\alpha \subset \mathbb{S}^n$ containing $\mathcal{U}'_\alpha$ such that its complement*

$$\mathbb{S}^n \smallsetminus B_\alpha$$

is connected. Furthermore, if there is a specified set A that is contained within one connected component of the disconnected set $\mathbb{S}^n \smallsetminus \mathcal{U}'_\alpha$, the open set B_α can be taken such that A is in the complement of B_α:

$$A \subset \mathbb{S}^n \smallsetminus B_\alpha.$$

Proof. Take a component $\mathcal{U}'_\alpha \subset \mathcal{U}'$ and the let L be a compact manifold of codimension 1 such that $\mathcal{U}'_\alpha \cong \mathbb{R} \times L$.

First, we take advantage of the Jordan-Brouwer separation theorem to fill in $\mathcal{U}'_\alpha$ and define the open set B_α. For each $t \in \mathbb{R}$, let L^t_α denote the embedded copy of L contained within $\mathcal{U}'_\alpha$ associated to the image of $\{t\} \times L$ under the diffeomorphism

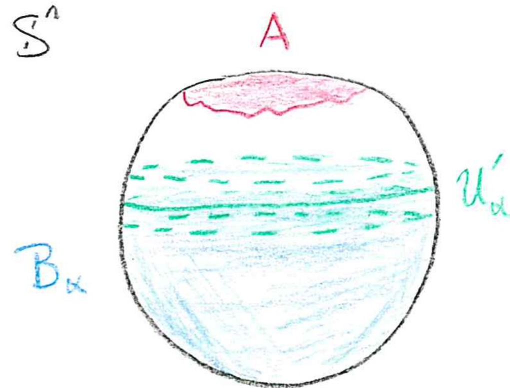

Figure 5.8: Simple example of B_α. Note that $\mathcal{U}'_\alpha \subset B_\alpha$.

$\mathbb{R} \times L \cong \mathcal{U}'_\alpha$. Note that, via this diffeomorphism, the leafs L^t_α form a partition of $\mathcal{U}'_\alpha$:

$$\mathcal{U}'_\alpha = \coprod_{t \in \mathbb{R}} L^t_\alpha.$$

By the Jordan-Brouwer separation theorem, each L^t_α bounds two disjoint, connected, open subsets U^t_α, V^t_α of $\mathbb{S}^n$:

$$\mathbb{S}^n \setminus L^t_\alpha = U^t_\alpha \coprod V^t_\alpha.$$

The set A is disjoint from L^t_α, so it resides in one of these components. Adopt the convention that A is contained in V^t_α for each $t \in \mathbb{R}$. Further, we establish a nesting of the sets U^t_α. For any s in $\mathbb{R}$, since L^s_α is connected, L^s_α is either in U^t_α or V^t_α. We will adopt the following labeling convention: if $s < t$, then L^s_α is contained in U^t_α. Thus, for $s < t$, since L^s_α separates the space, there is a nesting of open sets $U^s_\alpha \subset U^t_\alpha$, each of which has boundary diffeomorphic to L.

Define the open set

$$B_\alpha := \bigcup_{k \in \mathbb{Z}} U^k_\alpha.$$

First, we will note that $\mathcal{U}'_\alpha$ is contained within B_α. Indeed, for each $t \in \mathbb{R}$ and

the associated $L^t_\alpha \subset \mathcal{U}'_\alpha$, there is an integer $k \in \mathbb{Z}$ such that $t < k$ and $L^t_\alpha \subset U^k_\alpha$. As the collection of L^t_α partition $\mathcal{U}'_\alpha$ and each L^t_α is contained in a subset of the union, $\mathcal{U}'_\alpha \subset B_\alpha$.

Now, we argue that the complement of B_α is connected in $\mathbb{S}^n$. For each $k \in \mathbb{Z}$, the set $\mathbb{S}^n \setminus U^k_\alpha = L^k_\alpha \bigcup V^k_\alpha$ is closed and connected. For any pair of indices $k < k'$, the associated interior sets are nested $U^k_\alpha \subset U^{k'}_\alpha$. Thus, their complements are also nested:

$$\mathbb{S}^n \setminus U^{k'}_\alpha \subset \mathbb{S}^n \setminus U^k_\alpha.$$

So, since $\mathbb{S}^n$ is a normal space and the collection of compact and connected subsets $\{\mathbb{S}^n \setminus U^k_\alpha\}_{k\in\mathbb{Z}}$ is nested, the intersection of this collection,

$$\bigcap_{k\in\mathbb{Z}} \mathbb{S}^n \setminus U^k_\alpha = \mathbb{S}^n \setminus \bigcup_{k\in\mathbb{Z}} U^k_\alpha = \mathbb{S}^n \setminus B_\alpha,$$

is connected.

Finally, it is clear that A is contained in $\mathbb{S}^n \setminus B_\alpha$ as A is a subset of the complement of U^k_α for each $k \in \mathbb{Z}$.

$\square$

We will now argue that $\mathbb{S}^n \setminus \bigcup_{\alpha\in J} B_\alpha$ is connected. Here, J is the countable collection of indices numerating the connected components of $\mathcal{U}'$. Fix an enumeration of the set $\{\alpha_1, \alpha_2, \ldots\}$.

Recall that each B_α is defined as the union of the interiors U^k_α of the compact leafs L^k_α in the associated $\mathcal{U}'_\alpha$,

$$B_\alpha := \bigcup_{k\in\mathbb{Z}} U^k_\alpha.$$

Thus, by De Morgan's law, we can express the set that we will show is connected by

$$\mathbb{S}^n \setminus \bigcup_{\alpha \in J} B_\alpha = \bigcap_{k \in \mathbb{Z}} \left(\bigcap_{\alpha \in J} \mathbb{S}^n \setminus U_\alpha^k \right).$$

Observation 5.0.35. If there exists indices $\alpha, \alpha' \in J$ such that the intersection $B_\alpha \cap B_{\alpha'} \neq \varnothing$ is non-empty, then one of the open subsets is contained within the other, that is, $B_\alpha \subset B_{\alpha'}$ or $B_{\alpha'} \subset B_\alpha$. This is since each leaf separates the space and the components of $\mathcal{U}'$ have empty intersection. As such, the subset $\mathbb{S}^n \setminus \bigcup_{\alpha \in J} B_\alpha$ is equal to the subset where J is replaced by the subcollection of indices such that $B_\alpha \cap B_{\alpha'} = \varnothing$ for all pairs of indices α and α'. In the next proof, we will assume that J represents the subcollection of indices such that the collection $\{B_\alpha\}_{\alpha \in J}$ is pairwise disjoint.

Lemma 5.0.36. *Let $\{B_\alpha\}_{\alpha \in J}$ be the collection of open subsets defined in Lemma 5.0.34 and J be the countable set indexing the components of $\mathcal{U}'$. Then, the subset*

$$\mathbb{S}^n \setminus \bigcup_{\alpha \in J} B_\alpha = \bigcap_{k \in \mathbb{Z}} \left(\bigcap_{\alpha \in J} \mathbb{S}^n \setminus U_\alpha^k \right)$$

in $\mathbb{S}^n$ is connected.

Proof. First, we will argue that the subset

$$\bigcap_{i=1}^{N} \mathbb{S}^n \setminus U_{\alpha_i}^k \subset \mathbb{S}^n$$

is connected for any positive integer N. Suppose a separation $\{W_1, W_2\}$ exists. Each leaf $L_{\alpha_i}^k$ is connected. So, each leaf appears in exactly one of the two separating subsets, W_1 or W_2.

Consider the open sets

$$\left\{ W_1 \cup \left(\bigcup_{\substack{i \in \{1,\ldots,N\} \\ L^k_{\alpha_i} \subset W_1}} U^k_{\alpha_i} \right), W_2 \cup \left(\bigcup_{\substack{i \in \{1,\ldots,N\} \\ L^k_{\alpha_i} \subset W_2}} U^k_{\alpha_i} \right) \right\}. \tag{5.1}$$

Each subset is indeed open as we have been careful to include the interior $U^k_{\alpha_i}$ of a given leaf $L^k_{\alpha_i}$ with the subset of the original separation, W_1 or W_2, that contains that leaf.

These two open subsets of $\mathbb{S}^n$ form a two-term open cover of the n-sphere. Suppose there is an element p that is in the intersection of the two open subsets indicated in (5.1). The subsets W_1 and W_2 are disjoint from each other. Also, both W_1 and W_2 are disjoint from the subset $U^k_{\alpha_i}$ for any i. Then, there exists indices $i \neq j$ such that $p \in U^k_{\alpha_i} \cap U^k_{\alpha_j} \neq \varnothing$. Thus, the open subsets $B^k_{\alpha_i}$ and $B^k_{\alpha_j}$ have non-empty intersection. But, by Observation 5.0.35, the open subsets $B^k_{\alpha_i}$ and $B^k_{\alpha_j}$ have empty intersection. Thus, no such element p exists. Therefore, the two-term open cover defined in (5.1) is a separation of the n-sphere, a contradiction.

Thus, the compact subset $\cap_{i=1}^N \mathbb{S}^n \smallsetminus U^k_{\alpha_i}$ in $\mathbb{S}^n$ is connected for any positive integer N. Therefore, the compact subset of the normal space $\mathbb{S}^n$

$$\bigcap_{\alpha \in J} \mathbb{S}^n \smallsetminus U^k_\alpha \subset \mathbb{S}^n = \bigcap_{N=1}^{\infty} \left(\bigcap_{i=1}^{N} \mathbb{S}^n \smallsetminus U^k_{\alpha_i} \right)$$

is connected as it is the nested intersection of compact and connected subsets.

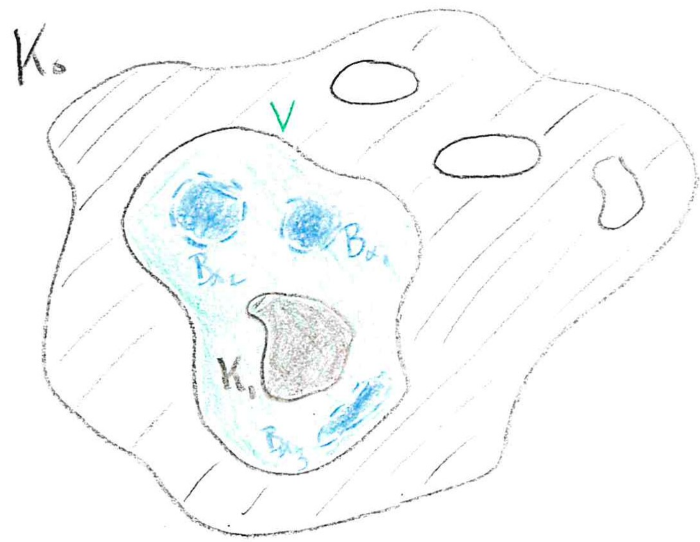

Figure 5.9: K_0 and K_1 in $\mathbb{S}^n$.

Finally, by similar logic, the compact subset of the normal space $\mathbb{S}^n$

$$\mathbb{S}^n \setminus \bigcup_{\alpha \in J} B_\alpha = \bigcap_{k \in \mathbb{Z}} \left(\bigcap_{\alpha \in J} \mathbb{S}^n \setminus U_\alpha^k \right)$$

is connected as it is the nested intersection of compact and connected subsets. This completes the proof.

□

Lemma 5.0.37. *Let $K_0, K_1 \subset \mathcal{K}'$ be disjoint, connected components of $\mathcal{K}'$. Then, there exists a compact leaf in the set $\mathcal{U}'$ that separates K_0 and K_1.*

Proof. Suppose, by way of contradiction, that there is no connected component of the set $\mathcal{U}'$ that separates the components K_0 and K_1.

Let V be the connected component of $\mathbb{S}^n \setminus K_0$ that contains K_1. Since K_1 is connected and does not intersect K_0, it is contained in exactly one such connected component.

Consider the union $K_0 \bigcup V \subset \mathbb{S}^n$. The subset is closed because its complement in $\mathbb{S}^n$ is the union of the remaining connected components of $\mathbb{S}^n \smallsetminus K_0$.

Also, as will be argued, $K_0 \bigcup V$ is connected. Suppose that there was a separation of $K_0 \bigcup V$ given by some open sets $W_1, W_2 \subset \mathbb{S}^n$, where $K_0 \subset W_1$ and $V \subset W_2$. Since the component K_0 does not intersect W_2, the separation can be rewritten as W_1 and $W_2 \cap V = V$. Now, consider the union of all components of $\mathbb{S}^n \smallsetminus K_0$ other than V with the open set W_1. The resulting open set contains the entirety of $\mathbb{S}^n \smallsetminus V$ and does not intersect V. Thus, this is a separation of the connected space $\mathbb{S}^n$, a contradiction. So, $K_0 \cup V$ is connected and closed.

Consider the collection of components of $\mathcal{U}'$ in the open set V. Index the components of the intersection $\mathcal{U}' \cap V$ by the countable indexing set $J_V \subset J$. Each $\mathcal{U}'_\alpha$ is diffeomorphic to the product of $\mathbb{R}$ with some codimension 1 compact submanifold L_α and thus, by the Jordan-Brouwer separation theorem, $\mathcal{U}'_\alpha$ separates $\mathbb{S}^n$.

As it was assumed that no component of $\mathcal{U}'_\alpha$ separates K_0 from K_1, by Lemma 5.0.34, for each $\alpha \in J_V$, there is an open set B_α containing the component $\mathcal{U}'_\alpha$ such that the union $K_0 \cup K_1$ is contained in the complement of B_α.

We will argue that the subset

$$K_0 \bigcup \left(V \smallsetminus \bigcup_{\alpha \in J_V} B_\alpha \right)$$

is connected and contained in K. This will yield a contradiction as the components K_0 and K_1, which were assumed to be disjoint connected components of K, are both contained in a single connected component of K.

Note that the subset $K_0 \bigcup (V \smallsetminus \bigcup_{\alpha \in J_V} B_\alpha)$ is contained in the connected subset $\mathbb{S}^n \smallsetminus \bigcup_{\alpha \in J} B_\alpha$ (Lemma 5.0.36). A separation of the former subset can be used to form a separation of the latter subset by including the other components of $\mathbb{S}^n \smallsetminus K_0$

Figure 5.10: Examples of trees.

and removing all of the open subsets B_α contained within. As this would be a contradiction, the subset $K_0 \bigcup (V \smallsetminus \bigcup_{\alpha \in J_V} B_\alpha)$ is connected.

Furthermore, since $\mathcal{U}'_\alpha \subset B_\alpha$ for each $\alpha \in J_V$, the closed set

$$K_0 \bigcup \left(V \smallsetminus \bigcup_{\alpha \in J_V} B_\alpha \right)$$

does not intersect $\mathcal{U}'$ and thus is contained in $\mathcal{K}'$. Since this set is also connected, it represents a connected component of $\mathcal{K}'$. But, this set contains K_0 and K_1. Since it was assumed that K_0 and K_1 were disjoint components of $\mathcal{K}'$, we reach a contradiction.

□

Definition 5.0.38. An element x of a connected topological space X is a *cut point* if $X \smallsetminus \{x\}$ is disconnected.

Definition 5.0.39. A connected, compact, Hausdorff, locally connected space X is a *tree* if, for any pair of distinct points $y, z \in X$, there exists a cut point $x \in X$ such that x separates y from z.

See [28] and [33] for background into the study of trees. See Figure 5.10 for examples of trees.

Lemma 5.0.40. *The quasi-leaf space $q(\mathbb{S}^n)$ is a tree.*

Proof. Since the space $\mathbb{S}^n$ is connected and compact, the quasi-leaf space $q(\mathbb{S}^n)$ is also connected and compact. It is then enough to show that for any quasi-leafs in $\mathbb{S}^n$, there exists a third quasi-leaf that separates them.

Let $L \subset \mathcal{U}'$ be a leaf of the foliation of $\mathcal{U}'$ by compact submanifolds of codimension 1 and A be any distinct quasi-leaf. By the Jordan-Brouwer separation theorem, L separates the space,

$$\mathbb{S}^n \setminus L = W_0 \coprod W_1.$$

The quasi-leaf A is connected and thus lies in exactly one of these components. Without loss of generality, assume that the quasi-leaf A is contained in W_0.

Let $\mathcal{U}'_L \subset \mathcal{U}'$ be the component of the set $\mathcal{U}'$ which contains the leaf L. By Proposition 5.0.11, there exists an open neighborhood V_L of L that is contained in the component $\mathcal{U}'_L$. There is an associated distributional diffeomorphism

$$\varphi^{X_L} : ((-\varepsilon, \varepsilon) \times L, (-\varepsilon, \varepsilon) \times TL) \xrightarrow{\cong} (V_L, \ker(Df)).$$

Suppose that the quasi-leaf A could be represented at the image of the compact leaf L under the map φ^{X_L} for some time $t \in (-\varepsilon, \varepsilon)$, that is, $A = \varphi_t^{X_L}(L)$. Then, the compact leaf $\varphi_{t/2}^{X_L}(L)$ separates A and L since $\{t/2\} \times L$ separates the leafs $\{t\} \times L$ and $\{0\} \times L$ in the product space $(-\varepsilon, \varepsilon) \times L$.

Suppose that the quasi-leaf A was not in the image of the diffeomorphism φ^{X_L}. Consider the sets $\varphi_{-\varepsilon/2}^{X_L}(L)$ and $\varphi_{\varepsilon/2}^{X_L}(L)$, which are each codimension 1 compact leafs that separate $\mathbb{S}^n$. One of these compact leafs separates the compact leaf L from the quasi-leaf A. Otherwise, the quasi-leaf A would be in the image of the restricted map $\varphi^{X_L}|_{(-\varepsilon/2, \varepsilon/2) \times L}$, a contradiction.

So, any quasi-leaf in $\mathbb{S}^n$ associated to a compact leaf of $\mathcal{U}'$ can be separated from any other quasi-leaf.

Take K_0 and K_1 to be disjoint components of $\mathcal{K}'$. By Lemma 5.0.37, there exists a component $\mathcal{U}'_\alpha \subset \mathcal{U}'$ that separates K_0 and K_1. Use the distributional isomorphism $\mathcal{U}'_\alpha \cong \mathbb{R} \times L_\alpha$ to select a compact leaf L. The compact leaf L is an equivalence class in $\mathbb{S}^n$ which separates K_0 from K_1.

Since any pair of quasi-leafs A_0 and A_1 can be separated in $\mathbb{S}^n$ by a compact leaf L, then there exists open subsets W_0 and W_1 such that $\mathbb{S}^n \smallsetminus L = W_0 \coprod W_1$ and $A_0 \subset W_0$ and $A_1 \subset W_1$. Thus, the subsets of the quasi-leaf space $q(W_0)$ and $q(W_1)$, which contain $q(A_0)$ and $q(A_1)$ respectively, are disjoint. These subsets are also open since $q^{-1}(q(W_0)) = W_0$ and $q^{-1}(q(W_1)) = W_1$. Therefore, the quasi-leaf space is Hausdorff.

Now, for a quasi-leaf $q(A) \in q(\mathbb{S}^n)$, let $U \subset q(\mathbb{S}^n)$ be an open neighborhood of $q(A)$. Then, the preimage $q^{-1}(U)$ is an open subset of $\mathbb{S}^n$ that contains the connected and compact subset A. Moreover, every quasi-leaf that has non-empty intersection with $q^{-1}(U)$ is contained within the open subset. Take the connected components $U' \subset q^{-1}(U)$ that contains the connected subset A. Then, the subset $q(U') \subset q(\mathbb{S}^n)$ is open as $q^{-1}(q(U')) = U'$. Since U' is connected, so is the image $q(U')$. Therefore, the quasi-leaf space is locally connected.

□

Smooth rank 1 maps from $\mathbb{S}^n$ are smoothly null-homotopic

Theorem 5.0.41. *A smooth map $f : \mathbb{S}^n \longrightarrow M$ of rank at most 1 is smoothly null-homotopic.*

Proof. By Observation 5.0.32, the map f continuously factors through the quasi-leaf space,

$$\begin{tikzcd}
\mathbb{S}^n \arrow[rr, "f"] \arrow[dr, "q"'] & & M. \\
& q(\mathbb{S}^n) \arrow[ur, "f_q"'] &
\end{tikzcd}$$

By Lemma 5.0.40, the quasi-leaf space $q(\mathbb{S}^n)$ is a tree. A result of Paulowich states that any tree is contractible [28]. So, there exists a continuous map,

$$h : [0,1] \times q(\mathbb{S}^n) \longrightarrow q(\mathbb{S}^n),$$

such that the map h_0 is the identity on the leaf space and h_1 is constant. Thus, there is a continuous homotopy,

$$H : [0,1] \times \mathbb{S}^n \xrightarrow{(\mathbb{1}_{[0,1]}, q)} [0,1] \times q(\mathbb{S}^n) \xrightarrow{h} q(\mathbb{S}^n) \xrightarrow{f_q} M,$$

such that there is an equality of maps $H_0 = f$ and the map H_1 is constant. Thus, the map f is continuously null-homotopic.

Via Whitney approximation theorem [35], the continuous null-homotopy H can be smoothed so that the map f is smoothly null-homotopic. $\square$

Higher horizontal homotopy groups of contact 3-manifolds

Corollary 5.0.42. *A horizontal map $f : \mathbb{S}^n \longrightarrow (M, \xi)$ into a contact 3-manifold is smoothly null-homotopic.*

Corollary 5.0.42 indicates why it is reasonable to assert that all such horizontal maps into a contact 3-manifold are horizontally null-homotopic.

Conjecture 5.0.43. *Fix a three-dimensional contact manifold (M, ξ). Then, for $n \geq 2$,*

$$\pi_n^{\mathsf{H}}(M, \xi) = 0.$$

Equivalently, this can be worded as follows: For $n \geq 2$*, given any contact map* $f : \mathbb{S}^n \longrightarrow (M, \xi)$*, there exists a contact map* $\tilde{f} : \mathbb{D}^{n+1} \longrightarrow (M, \xi)$ *that is an extension of* f*.*

$$\begin{tikzcd} \mathbb{S}^n \arrow[r, "f"] \arrow[d, hook, "\partial"] & (M,\xi) \\ \mathbb{D}^{n+1} \arrow[ur, dotted, "\tilde{f}"'] & \end{tikzcd}$$

The smooth null-homotopy indicated by in Theorem 5.0.41 is guaranteed by the Whitney approximation Theorem. Unfortunately, this distorts the image of the map in the target. Thus, there is no guarantee that the resulting smooth homotopy is a horizontal map.

HIGHER LIPSCHITZ HOMOTOPY GROUPS OF CONTACT 3-MANIFOLDS

We now transition from considering horizontal homotopy groups to considering Lipschitz homotopy groups.

Lipschitz homotopy groups are a less natural, but more fruitful, tool that horizontal homotopy groups. Horizontal homotopy groups are a natural tool for studying contact manifolds as these groups capture the result of probing the space with smooth maps without introducing further structure on the space. Alas, π_n^{H} can be difficult to calculate, as is illustrated in this dissertation when $n > 1$. Though introducing a compatible metric adds more structure to the space, and thus detracts from the pureness of the probing, the Lipschitz homotopy groups of a contact manifold are easier to access as 'smooth' can be too strict of a condition to work with. By weakening 'smooth' to 'Lipschitz' and π_n^{H} to π_n^{Lip}, some calculations become easier. For example, π_n^{Lip} is trivial for $n > 1$. See [6] and [34] for papers inspecting the differences between these homotopy groups.

The aim of this chapter is to calculate the higher Lipschitz homotopy groups of any contact 3-manifold, showing that these groups are all trivial. This result has already been achieved in the case where the contact 3-manifold in question is the first Heisenberg group, $\mathbb{H}^1$ [34]. In fact, they proved a more general result, essentially showing that any purely 2-unrectifiable metric space has trivial higher Lipschitz homotopy groups.

Theorem 6.0.1. ***(Wenger and Young, Theorem 5 in [34])*** *Let X be a quasi-convex metric space with $\pi_1^{Lip}(X) = 0$. Let furthermore Y be a purely 2-unrectifiable metric space. Then every Lipschitz map from X to Y factors through a metric tree.*

The n-sphere $X = \mathbb{S}^n$ with the standard Riemannian metric is quasi-convex and simply connected for $n \geq 2$. So, all Lipschitz mappings from $\mathbb{S}^n$ to a purely

2-unrectifiable space factor through a metric tree, which is a Lipschitz contractible space. Thus, all such mappings are Lipschitz null-homotopic and the higher Lipschitz homotopy groups are trivial. As such, to establish the desired calculation, it is then enough to show that a contact 3-manifold, endowed with a sub-Riemannian structure, is purely 2-unrectifiable.

This will prove true as any contact 3-manifold is locally modeled on a purely 2-unrectifiable space. By a Theorem of Darboux, contact 3-manifolds locally look like copies of $\mathbb{H}^1$ [13] and $\mathbb{H}^1$ is a purely 2-unrectifiable space [1]. As will be shown, the distributional embeddings guaranteed by Darboux can be restricted to be biLipschitz with respect to the associated Carnot-Carathéodory metrics and carry this metric condition on $\mathbb{H}^1$ to the contact manifold.

This will be shown by inspecting the interplay of distributional maps and the Carnot-Carathéodory lengths of paths. First, we will show that the length of the image of a horizontal path under distributional map, which again is a horizontal path, is bounded. Thus, the distributional embedding guaranteed by the theorem of Darboux can only distort lengths of paths, and thus distances between points, by a manageable amount.

Next, we will account for subsets of contact manifolds not-in-general being geodesically convex. We will show that for any open ball in a sub-Riemannian manifold, that there is a bounded open set containing the ball in which distances between points in the ball can be well-approximated by horizontal paths that remain in the new bounded open set. These tools will be enough to restrict a distributional embedding to a neighborhood such that the map is also biLipschitz with respect to the Carnot-Carathéodory metrics.

Distributional maps and Carnot-Carathéodory lengths of paths

As was the case with manifolds with a distribution, distributional mappings will be the primary means of mapping between sub-Riemannian manifolds. In the case that there are sub-Riemannian manifolds (M, ξ, g) and (M', ξ', g') and a distributional map $f : (M, \xi) \longrightarrow (M', \xi')$, we will denote the map by $f : (M, \xi, g) \longrightarrow (M', \xi', g')$.

As noted in Lemma 2.0.12, the image of a horizontal path under a distributional map is horizontal. The next result articulates a sense in which lengths are scaled by pushforward via a distributional map.

This will be desirable as we will restrict the distributional embedding guaranteed by the Theorem of Darboux in order to produce a biLipschitz map. As the Carnot-Carathéodory metric is defined as in terms of lengths of horizontal curves, it is important to know how much distortion can be caused by a distributional embedding in order to compare the metrics.

Lemma 6.0.2. *Let $f : (M, \xi, g) \longrightarrow (M', \xi', g')$ be a distributional map between sub-Riemannian manifolds. Let $A \subset M$ be a compact set. Then, there exists $B > 0$ such that, for any horizontal path $\gamma : [a, b] \longrightarrow (M, \xi)$ mapping into A, the length of the horizontal path $f \circ \gamma$ is bounded:*

$$l^{M'}(f \circ \gamma) \leq B \; l^{M}(\gamma).$$

Before proving Lemma 6.0.2, we will need to take a detour to consider the operator norm on the derivative of a distributional map. As will be argued, the operator norm on the derivative of a fixed distributional map is bounded on compact sets. This will yield the desired B in Lemma 6.0.2.

Definition 6.0.3. For a distributional map between sub-Riemannian manifolds $f :$

$(M, \xi, g) \longrightarrow (M', \xi', g')$, the *(operator) norm* of the derivative at $p \in M$ is

$$\|D_p f\| := \sup_{\substack{v \in \xi_p \\ g(v,v)=1}} \sqrt{g'(D_p f(v), D_p f(v))}.$$

Remark 6.0.4. This is the operator norm of the linear map $D_p f : \xi_p \longrightarrow \xi'_{f(p)}$ with respect to the associated inner products. Indeed, the vector space ξ_p with the sub-Riemannian metric is a Hilbert space, as is $\xi'_{f(p)}$. As with the standard operator norm, $\|D_p f\|$ provides a useful bound on the magnitude of the image of any vector $v \in \xi_p$ under the mapping $D_p f$:

$$\sqrt{g'(D_p f(v), D_p f(v))} \leq \|D_p f\| \sqrt{g(v, v)}.$$

For a fixed distributional map f, consider the map between sets,

$$\|D_{(-)} f\| : M \longrightarrow \mathbb{R}$$

$$p \longmapsto \|D_p f\|.$$

This map, as will be verified, is upper semi-continuous. Upper semi-continuity on a compact set is enough the guarantee an upper bound on the mapping.

Definition 6.0.5. A map $G : X \longrightarrow \mathbb{R}$ from topological space X is *upper semi-continuous* at $x_0 \in X$ if for all $\varepsilon > 0$ there exists an open neighborhood $U \subset X$ of x_0 such that, for all $x \in U$,

$$G(x) \leq G(x_0) + \varepsilon.$$

If G is upper semi-continuous at x_0 all points $x_0 \in X$, G is *upper semi-continuous.*

In order to guarantee that $\|D_{(-)} f\|$ is upper semi-continuous, we take advantage of the map being defined as a supremum of a continuous map over a fiber bundle whose fibers are compact.

First, we indicate the fiber bundle. For sub-Riemannian manifold (M, ξ, g), let $\mathbb{S}^{\xi}M \subset \xi \subset TM$ denote the fiber bundle over M whose fiber over $p \in M$ is the collection of vectors in ξ_p with unit length:

$$\mathbb{S}^{\xi}_{p}M := \{v_p \in \xi_p : g(v_p, v_p) = 1\}.$$

In other words, the fiber over a point is the unit sphere in the fiber of ξ with respect to g. The fiber bundle $\mathbb{S}^{\xi}M$ is referred to as the *unit sphere bundle of* ξ *over* M.

Now, we indicate the continuous map.

$$\begin{aligned} F : \mathbb{S}^{\xi}M &\longrightarrow \mathbb{R} \\ v_p &\longmapsto \sqrt{g'(D_p f(v_p), D_p f(v_p))}. \end{aligned}$$

The map F is indeed well-defined as, for any $v_p \in \mathbb{S}^{\xi}M$, since f is a distributional map, the vector $D_p f(v_p)$ is in $\xi'_{f(p)}$. Thus, we can take its norm with respect to g', which is smoothly-varying over the distribution ξ'.

Since F is defined on a fiber bundle, this continuous map will have domain that is locally a product of Euclidean space and a sphere. The supremum at each point of M is taken over the fiber in the sphere bundle, which is compact. This will be used to get upper semi-continuity.

Lemma 6.0.6. *Let X and K be topological spaces and assume that K is compact. Let $F : X \times K \longrightarrow \mathbb{R}$ be a continuous map. Then the map*

$$\begin{aligned} X &\longrightarrow \mathbb{R} \\ x &\longmapsto \sup_{y \in K} F(x, y). \end{aligned}$$

is upper semi-continuous.

Proof. Take $x_0 \in X$. Let $\varepsilon > 0$ be given. Since F is continuous, for all $y_0 \in K$, there exists an open neighborhood $U_0 \times V_0 \subset X \times K$ of (x_0, y_0) such that for any $(x, y) \in U_0 \times V_0$,

$$F(x_0, y_0) - \varepsilon < F(x, y) < F(x_0, y_0) + \varepsilon.$$

As $\{x_0\} \times K$ is a compact set, there exists a finite cover by such neighborhoods. That is, there exists a finite collection of open sets $\{U_i \times V_i\}_{i=1}^n$ and points $y_i \in V_i$ such that the collection of V_i cover K and, for all $(x, y) \in U_i \times V_i$,

$$F(x_0, y_i) - \varepsilon < F(x, y) < F(x_0, y_i) + \varepsilon.$$

Let $L \in \mathbb{R}$ be an upper bound for $\{F(x_0, y)\}_{y \in K}$. Note that such an upper bound is guaranteed to exist as $F(x_0, -) + \varepsilon : K \longrightarrow \mathbb{R}$ is a continuous map with compact domain. The collection of points of K generated from the finite cover is a subset of K; $\{y_1, \ldots, y_n\} \subset K$. Thus, L is also an upperbound for

$$\{F(x_0, y_i) + \varepsilon\}_{i=1}^n \subset \{F(x_0, y) + \varepsilon\}_{y \in K}.$$

Consider the open set $\bigcap_{i=1}^n U_i \subset X$. Take a point $x \in \bigcap_{i=1}^n U_i$. For any $y \in K$, there is an index i such that $(x, y) \in U_i \times V_i$ and

$$F(x, y) < F(x_0, y_i) + \varepsilon.$$

Thus, L is also an upper bound for $\{F(x, y)\}_{y \in K}$. So, any upper bound for $\{F(x_0, y) + \varepsilon\}_{y \in K}$ is also an upper bound for $\{F(x, y)\}_{y \in K}$. In particular, $L = \sup_{y \in K} F(x_0, y) + \varepsilon$. Therefore,

$$\sup_{y \in K} F(x, y) \leq \sup_{y \in K} F(x_0, y) + \varepsilon$$

for all $x \in \bigcap_{i=1}^{n} U_i$ and the map in question is upper semi-continuous.

$\square$

Proposition 6.0.7. *Let $f : (M, \xi, g) \longrightarrow (M', \xi', g')$ be a distributional map between sub-Riemannian manifolds. Let $A \subset M$ be a compact set. Then there exists $B > 0$ such that*

$$\|D_p f\| \leq B \text{ for all } p \in A.$$

Proof. First, we note that the map

$$\begin{aligned} F : \mathbb{S}^{\xi} M &\longrightarrow \mathbb{R} \\ v_p &\longmapsto \sqrt{g'(D_p f(v_p), D_p f(v_p))}. \end{aligned}$$

is continuous. We then take advantage of the local product structure of $\mathbb{S}^{\xi} M$ and Lemma 6.0.6 to see that

$$\begin{aligned} \|D_{(-)} f\| : M &\longrightarrow \mathbb{R} \\ p &\longmapsto \sup_{v_p \in \mathbb{S}^{\xi}_p M} \sqrt{g'(D_p f(v_p), D_p f(v_p))}. \end{aligned}$$

is upper semi-continuous. Finally, as A is compact and upper semi-continuous maps on compact sets have an upper bound, the proof will be complete.

Indeed, the initial map is continuous as it can be written as a composition of continuous maps:

$$\begin{tikzcd} \mathbb{S}^{\xi} M \arrow[r, hook] \arrow[rrrr, bend right, "F"'] & \xi \arrow[r, "Df"] & \xi' \arrow[r, "\Delta"] & \xi' \times_{M'} \xi' \arrow[r, "\sqrt{g'}"] & \mathbb{R}. \end{tikzcd}$$

The diagonal map $\Delta : \xi' \longrightarrow \xi' \times_{M'} \xi'$ is the unique continuous map guaranteed by the universal property of pullback satisfying the following diagram:

$$\begin{tikzcd}
\xi' \arrow[dr, dashed, "\exists!", "\Delta"'] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\
& \xi' \times_{M'} \xi' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \xi' \arrow[d] \\
& \xi' \arrow[r] & M'.
\end{tikzcd}$$

Let $p_0 \in M$ and assume ξ is rank n. The fiber over any point in $\mathbb{S}^{\xi}M$ is diffeomorphic to $\mathbb{S}^{n-1}$ and, as $\mathbb{S}^{\xi}M \longrightarrow M$ is a fiber bundle, there exists trivializing open neighborhood $p_0 \in U \subset M$ such that

$$\begin{tikzcd}
U \times \mathbb{S}^{n-1} \arrow[rr, "\psi", "\cong"'] \arrow[dr] & & \mathbb{S}^{\xi}U \arrow[dl] \\
& U. &
\end{tikzcd}$$

Since ψ is continuous, so is $F \circ \psi : U \times \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$. As $\mathbb{S}^{n-1}$ is compact, by Lemma 6.0.6, the map

$$G : U \longrightarrow \mathbb{R}$$

$$p \longmapsto \sup_{v \in \mathbb{S}^{n-1}} F \circ \psi(p, v)$$

is upper semi-continuous. Now, $\psi(p, -) : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{\xi}_{p}M$ is a bijection between the two indexing sets that the supremum are defined over and the following diagram commutes:

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[d, "\psi(p,-)"'] \arrow[dr, "\sqrt{g'(D_pf(\psi(p,-)),D_pf(\psi(p,-))}"] & \\
\mathbb{S}^{\xi}_{p}M \arrow[r, "\sqrt{g'(D_pf(-),D_pf(-))}"'] & \mathbb{R}.
\end{tikzcd}$$

Thus,

$$\begin{aligned}
\|D_p f\| &= \sup_{v_p \in \mathbb{S}^{\xi}_p M} \sqrt{g'(D_p f(v_p), D_p f(v_p))} \\
&= \sup_{v \in \mathbb{S}^{n-1}} \sqrt{g'(D_p f(\psi(p,v)), D_p f(\psi(p,v))} \\
&= G(p)
\end{aligned}$$

and $\|D_{(-)} f\|$ is upper semi-continuous at $p_0 \in M$.

Since p_0 was an arbitrary element of M, the map $\|D_{(-)} f\|$ is upper semi-continuous on all of M. In particular, the map is upper semi-continuous on the compact set A. Thus, there exists upper bound B such that, for all $p \in A$,

$$\|D_p f\| \leq B.$$

□

We are now ready to prove Lemma 6.0.2.

Proof. Let γ be an horizontal map into (M, ξ) such that its image lies in the compact set A. As γ is a horizontal path and f is a distributional map, by Lemma 2.0.12, the composition $f \circ \gamma$ is a horizontal path in (M', ξ'). So, its Carnot-Carathéodory length is defined. The following string of inequalities uses the definition of Carnot-Carathéodory length, chain rule, Remark 6.0.4, and Proposition 6.0.7.

$$\begin{aligned}
l^{M'}(f \circ \gamma) &= \int_a^b \sqrt{g'\left(\frac{d}{dt}(f \circ \gamma(t)), \frac{d}{dt}(f \circ \gamma(t))\right)}\, dt \\
&= \int_a^b \sqrt{g'\left(D_{\gamma(t)} f(\dot\gamma(t)), D_{\gamma(t)} f(\dot\gamma(t))\right)}\, dt \\
&\le \int_a^b \|D_{\gamma(t)} f\| \sqrt{g\left(\dot\gamma(t), \dot\gamma(t)\right)}\, dt \\
&\le B \int_a^b \sqrt{g\left(\dot\gamma(t), \dot\gamma(t)\right)}\, dt \\
&= B\, l^M(\gamma).
\end{aligned}$$

□

BiLipschitz Darboux Theorem

As a consequence of the Theorem of Darboux, all contact 3-manifolds are locally modeled on $\mathbb{H}^1$, which, when endowed with the metric $d_{CC}^{\mathbb{H}^1}$, is purely 2-unrectifiable [1]. In order to relay this metric quality on $\mathbb{H}^1$ to a contact 3-manifold (M, ξ), we require an adjustment of the Theorem of Darboux. It will be shown that the distributional embeddings of $\mathbb{H}^1$ into (M, ξ) guaranteed by Darboux can be restricted such that they are biLipschitz with respect to the associated Carnot-Carathéodory metrics.

It is worth noting that it is not immediate that the distributional embeddings guaranteed by Darboux are locally biLipschitz with respect to Carnot-Carathéodory metrics. The distributional embeddings are smooth and thus, assuming there are Riemannian metrics on the associated manifolds, locally biLipschitz with respect to the path metrics. But, these path metrics are not necessarily biLipschitz equivalent to the Carnot-Carathéodory metrics. Indeed, it is known for any sub-Riemannian manifold that these two metrics are not biLipschitz equivalent (Theorem

2.10 in [24]).

Thus, to guarantee that these distributional embeddings are taken to be locally biLipschitz, or even locally Lipschitz, we must better understand the Carnot-Carathéodory metric, in particular, where horizontal curves approximating the distance between two points live. As the Carnot-Carathéodory metric is defined in terms of lengths of horizontal curves, it is desirable to know how a distributional map can distort these lengths. Lemma 6.0.2 yields such a tool for bounding lengths of paths that live in a compact set. Choosing an open subset of the domain that is bounded then becomes the focus.

As Lemma 6.0.2 yields a bound for horizontal paths that remain in a compact set, it is important that the bounded open set contains horizontal paths that well-approximate the Carnot-Carathéodory distance between some set of points. In practice, we cannot expect that the set containing the points and the set containing the horizontal paths to be equal. Given an arbitrary open subset of a contact manifold, it is unlikely that it is *geodetically convex*, i.e., contains all length-minimizing horizontal paths between all of its points. Indeed, it is known that the only geodetically convex open subset of $\mathbb{H}^1$ is itself [25]. This is in contrast to the Euclidean case where open balls are convex.

So, we should not expect to be able to well-approximate Carnot-Carathéodory distance between points in a given bounded open set via horizontal paths that remain in the open set. Rather, given a bounded open set, there is a larger but still bounded open set in which the Carnot-Carathéodory distance between points in the former open set can be well-approximated via horizontal paths that live in the latter.

Lemma 6.0.8. *Let (M, ξ, g) be a sub-Riemannian manifold. Consider the open ball*

$B^M_{CC}(p,r) \subset M$. The set

$$U(p,r) := \bigcup_{q \in B^M_{CC}(p,r)} B^M_{CC}(q, 2r)$$

satisfies the following properties:

1. *$U(p,r)$ is open and bounded with respect to d^M_{CC} and contains $B^M_{CC}(p,r)$.*

2. *Let $x, y \in B^M_{CC}(p,r)$ and let $0 < \varepsilon < 2r - d^M_{CC}(x,y)$. Then there exists a horizontal path*

$$\begin{tikzcd} {[0,1]} \arrow[rr, "\gamma"] \arrow[dr] & & (M,\xi) \\ & U(p,r) \arrow[ur, hook] & \end{tikzcd}$$

such that

$$d^M_{CC}(x,y) < l^M(\gamma) \leq d^M_{CC}(x,y) + \varepsilon < 2r.$$

Property *(2)* is what is meant by Carnot-Carathéodory distance between two points being well-approximated via horizontal paths.

Proof. (1) $U(p,r)$ is the union of open balls and is thus an open set. To see that $U(p,r)$ is bounded, consider the distance between p and an arbitrary element $x \in U(p,r)$. By definition of $U(p,r)$, there exists $q \in B^M_{CC}(p,r)$ such that $x \in B^M_{CC}(q, 2r)$. By triangle inequality,

$$d^M_{CC}(x,p) \leq d^M_{CC}(x,q) + d^M_{CC}(q,p) < 2r + r = 3r$$

and $x \in B^M_{CC}(p, 3r)$. So, $U(p,r)$ is contained in the open ball $B^M_{CC}(p,3r)$ and is therefore bounded.

Also, let $q \in B^M_{CC}(p,r)$. As q is contained in the open ball $B^M_{CC}(q,2r)$, by choice of $U(p,r)$, q is in $U(p,r)$. Since q was an arbitrary element of $B^M_{CC}(p,r)$, $B^M_{CC}(p,r) \subset U(p,r)$.

(2) Before verifying the existence of such a horizontal path, note that there is a bound on how far points in $B^M_{CC}(p,r)$ can be from one another. Let $x,y \in B^M_{CC}(p,r)$. By triangle inequality,

$$d^M_{CC}(x,y) \le d^M_{CC}(x,p) + d^M_{CC}(p,y) < r + r = 2r.$$

Let $0 < \varepsilon < 2r - d^M_{CC}(x,y)$. By the infimum definition of d^M_{CC}, there exists a horizontal path $\gamma : [0,1] \longrightarrow (M,\xi)$ such that $\gamma(0) = x$, $\gamma(1) = y$, and

$$d^M_{CC}(x,y) < l^M(\gamma) \le d^M_{CC}(x,y) + \varepsilon < 2r.$$

It remains to be verified that γ maps into $U(p,r)$. It is sufficient to see that γ maps into $B^M_{CC}(x,2r) \subset U(p,r)$.

Take $t \in [0,1]$. Since restricting a horizontal path yields another horizontal path, $\gamma|_{[0,t]}$ is a horizontal path in (M,ξ) connecting x and $\gamma(t)$. By the infimum definition of d^M_{CC}, the Carnot-Carathéodory distance between x and $\gamma(t)$ is no more than the length of this restriction; $d^M_{CC}(x,\gamma(t)) \le l^M(\gamma|_{[0,t]})$.

As noted in Lemma 2.0.53, the length of $\gamma|_{[0,t]}$ is no more than the length of γ. Thus,

$$d^M_{CC}(x,\gamma(t)) \le l^M(\gamma|_{[0,t]}) \le l^M(\gamma) \le d^M_{CC}(x,y) + \varepsilon < 2r$$

and $\gamma(t) \in B^M_{CC}(x,2r)$.

□

So, given an open ball with respect to the Carnot-Carathéodory metric, there

is a bounded open set that contains all horizontal paths that well-approximate the Carnot-Carathéodory distance between points in the ball. We will use the bound guaranteed by Lemma 6.0.2 on this larger bounded set to guarantee that distributional embeddings are locally Lipschitz.

Lemma 6.0.9. *Let (M,ξ) and (M',ξ') be contact manifolds and $U \subset M'$ be open. Suppose that for a point $p \in U$, there exists a radius $R > 0$ such that the closed ball $\overline{B^{M'}_{CC}(p,R)}$ is compact and is contained in U. For any distributional embedding $\varphi : (U,\xi'|_U) \hookrightarrow (M,\xi)$, there exists a neighborhood $p \in U' \subset \overline{B^{M'}_{CC}(p,R)}$ such that*

$$\varphi|_{U'} : (U',\xi'|_{U'}) \hookrightarrow (M,\xi)$$

is Lipschitz.

Proof. Fix a point $p \in U$ and a distributional embedding $\varphi : (U,\xi'|_U) \hookrightarrow (M,\xi)$. Let $U' = B^{M'}_{CC}(p,R/4)$.

Let $q,q' \in U'$. By Lemma 6.0.8, there exists open, bounded subset $U(p,R/4) \subset M'$ containing U' in which the Carnot-Carathéodory distance between q and q' can be well-approximated by lengths of horizontal paths in $U(p,R/4)$. Also by Lemma 6.0.8, the open subset $U(p,R/4) \subset B^{M'}_{CC}(p,3R/4) \subset U$ is contained in U.

Let $\varepsilon > 0$ be given. Then, there exists horizontal path $\gamma_\varepsilon : [0,1] \longrightarrow U(p,R/4)$ connecting q and q' such that

$$l^{M'}(\gamma_\varepsilon) \leq d^{M'}_{CC}(q,q') + \varepsilon.$$

Now, since $U(p,R/4) \subset \overline{B^{M'}_{CC}(p,R)}$, the open subset $U(p,R/4)$ is contained in a compact set. By Proposition 6.0.7, there exists $B > 0$ such that $\|D_{p'}\varphi\| < B$ for all

$p' \in U(p, R/4)$. As φ is a distributional map, by Lemma 6.0.2,

$$l^M(\varphi \circ \gamma_\varepsilon) \le B\ l^{M'}(\gamma_\varepsilon).$$

Since $\varphi \circ \gamma_\varepsilon$ is a horizontal path in M connecting the points $\varphi(q)$ and $\varphi(q')$, by the infimum definition of the metric, $d^M_{CC}(\varphi(q), \varphi(q')) \le l^M(\varphi \circ \gamma_\varepsilon)$. Stringing these inequalities together,

$$d^M_{CC}(\varphi(q), \varphi(q')) \le l^M(\varphi \circ \gamma_\varepsilon) \le B\ l^{M'}(\gamma_\varepsilon) \le B\ (d^{M'}_{CC}(q, q') + \varepsilon).$$

As ε can be taken to be arbitrarily small, $d^M_{CC}(\varphi(q), \varphi(q')) \le B\ d^{M'}_{CC}(q, q')$ and φ is Lipschitz on U'.

$\square$

This strategy can be used to guarantee that any distributional embedding is locally biLipschitz with respect to the Carnot-Carathéodory metrics. On the image of such a distributional embedding, there is an inverse that is also a distributional embedding. Use Lemma 6.0.9 to show that it too is locally Lipschitz and the original map is locally biLipschitz.

Lemma 6.0.10. *Let (M, ξ) and (M', ξ') be contact manifolds and $U \subset M'$ be open. Suppose that for a point $p \in U$, there exists a radius $R > 0$ such that the closed ball $\overline{B^{M'}_{CC}(p, R)}$ is compact and is contained in U. For any open distributional embedding $\varphi : (U, \xi'|_U) \hookrightarrow (M, \xi)$, there exists a neighborhood $p \in U'' \subset \overline{B^{M'}_{CC}(p, R)}$ such that*

$$\varphi|_{U''} : (U'', \xi'|_{U''}) \hookrightarrow (M, \xi)$$

is biLipschitz.

Proof. Let $p \in U$ be such a point in U and let $\varphi : (U, \xi'|_U) \hookrightarrow (M, \xi)$ be an open distributional embedding. By Lemma 6.0.9, there exists an open bounded neighborhood $p \in U' \subset \overline{B^{M'}_{CC}(p,R)}$ such that the restriction $\varphi|_{U'}$ is Lipschitz.

Now, since the map φ is invertible on all of $\varphi(U)$,

$$\varphi^{-1}|_{\varphi(U')} : (\varphi(U'), \xi|_{\varphi(U')}) \hookrightarrow (U', \xi|_{U'})$$

is also an open distributional embedding. Furthermore, the point $\varphi(p)$ is contained in the image of the closed ball of radius R:

$$\varphi(p) \in \varphi(U') \subset \varphi\left(\overline{B^{M'}_{CC}(p,R)}\right).$$

As the map φ is a diffeomorphism, the subset $\varphi\left(\overline{B^{M'}_{CC}(p,R)}\right)$ is compact in M. Also, the image $\varphi(U')$ is contained in the previously mentioned compact set and is an open neighborhood of the point $\varphi(p)$. Thus, there is a positive radius $R' > 0$ such that the open ball $B^{M}_{CC}(p,R')$ is contained in the open subset $\varphi(U')$. Additionally, the closed ball $\overline{B^{M}_{CC}(p,R')}$ is compact as it is contained in the compact subset $\overline{B^{M'}_{CC}(p,R)}$.

Again by Lemma 6.0.9, there exists an open neighborhood $\varphi(p) \in V \subset \overline{B^{M'}_{CC}(p,R)}$ on which φ^{-1} is Lipschitz.

Then, $U'' = \varphi^{-1}(V) \subset U'$ is an open set on which $\varphi|_{U''}$ is Lipschitz (since it is Lipschitz on U') and invertible. Since the inverse is also Lipschitz,

$$\varphi|_{U''} : (U'', \xi'|_{U''}) \xrightarrow{\cong} (\varphi(U''), \xi|_{\varphi(U'')})$$

is biLipschitz.

$\square$

With this more general result established, a biLipschitz Darboux Theorem is an

immediate corollary.

Corollary 6.0.11. *[BiLipschitz Darboux Theorem] Let (M, ξ) be a contact manifold. For every $p \in M$, there exists an open biLipschitz distributional embedding*

$$\varphi : (V, \xi^{std}|_V) \hookrightarrow (M, \xi)$$

where $V \subset \mathbb{H}^n$ is an open neighborhood of the origin and $\varphi(0) = p$.

Such a neighborhood $\varphi(V)$, along with the associated biLipschitz distributional embedding φ, will be referred to as a *biLipschitz Darboux neighborhood.*

Proof. For any point $p \in M$, there exists an open distributional embedding of $\mathbb{H}^n$,

$$\varphi : \mathbb{H}^n \hookrightarrow (M, \xi)$$

where $\varphi(0) = p$ (Theorem 2.0.36). Take the open subset U to be the entirety of $\mathbb{H}^n$. Since $\mathbb{H}^n$ is complete, any closed ball is compact. Lemma 6.0.10 guarantees that there is some open neighborhood $p \in V \subset \mathbb{H}^n$ such that $\varphi|_V$ is biLipschitz. □

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Contact 3-Manifolds are Purely 2-Unrectifiable

We are now ready to prove the main result of this chapter. As contact 3-manifolds are locally modeled by $\mathbb{H}^1$, they can be described as the union of biLipschitz Darboux neighborhoods. Since $\mathbb{H}^1$ is purely 2-unrectifiable, we will show that the union of these neighborhoods is as well.

Theorem 6.0.12. *Any contact 3-manifold (M, ξ) endowed with the Carnot-Carathéodory metric is purely 2-unrectifiable.*

Proof. Construct a cover of (M, ξ) by biLipschitz Darboux neighborhoods. By Corollary 6.0.11, each point in (M, ξ) has a biLipschitz Darboux neighborhood. M can be covered by such neighborhoods and, since M is a manifold, it can be reduced to a countable cover.

Let $\{\varphi_\alpha : (V_\alpha, \xi^{std}|_{V_\alpha}) \hookrightarrow (M, \xi)\}_{\alpha \in J}$ denote a countable collection of open biLipschitz distributional embeddings where $V_\alpha \subset \mathbb{H}^1$ is open for each $\alpha \in J$, such that $\{\varphi_\alpha(V_\alpha)\}_{\alpha \in J}$ is a countable cover of M.

Let $f : A \longrightarrow M$ be a Lipschitz map whose domain $A \subset \mathbb{R}^2$ is a Borel set. To verify that (M, ξ) is purely 2-unrectifiable with respect to d_{CC}^M, it is enough to show that $\mathcal{H}^2(\operatorname{Im} f) = 0$.

Fix an $\alpha \in J$ and consider f restricted to $f^{-1}(\varphi_\alpha(V_\alpha))$. By Corollary 6.0.11, $\varphi_\alpha^{-1} : \varphi_\alpha(V_\alpha) \longrightarrow V_\alpha$ is a Lipschitz map. As $f|_{f^{-1}(\varphi_\alpha(V_\alpha))}$ maps into $\varphi_\alpha(V_\alpha)$,

$$\varphi_\alpha^{-1} \circ f|_{f^{-1}(\varphi_\alpha(V_\alpha))} : f^{-1}(\varphi_\alpha(V_\alpha)) \longrightarrow V_\alpha$$

is defined and is Lipschitz;

$$\begin{tikzcd}
A \arrow[r, "f"] & M \\
 & \varphi_\alpha(V_\alpha) \arrow[u, hook] \\
f^{-1}(\varphi_\alpha(V_\alpha)) \arrow[uu, hook] \arrow[ur, "f|_{f^{-1}(\varphi_\alpha(V_\alpha))}"] \arrow[r, "\varphi_\alpha^{-1} \circ f|"] & V_\alpha. \arrow[u, "\cong"', "\varphi_\alpha"]
\end{tikzcd}$$

As $\varphi_\alpha(V_\alpha) \subset M$ is open and f is continuous, $f^{-1}(\varphi_\alpha(V_\alpha)) \subset A$ is an open subset of a Borel set and is thus Borel.

Since $\mathbb{H}^1$ is purely 2-unrectifiable [1], $\mathcal{H}^2(\operatorname{Im}(\varphi_\alpha^{-1} \circ f|_{f^{-1}(\varphi_\alpha(V_\alpha))})) = 0$. As φ_α is

Lipschitz, by Lemma 2.0.68,

$$\mathcal{H}^2(\mathrm{Im}(f|_{f^{-1}(\varphi_\alpha(V_\alpha))})) = \mathcal{H}^2(\varphi_\alpha(\mathrm{Im}(\varphi_\alpha^{-1} \circ f|_{f^{-1}(\varphi_\alpha(V_\alpha))}))) = 0.$$

Now, note that $\mathrm{Im}\, f = \bigcup_{\alpha \in J} \mathrm{Im}(f|_{f^{-1}(\varphi_\alpha(V_\alpha))})$. By subadditivity of the outer measure $\mathcal{H}^2$,

$$0 \leq \mathcal{H}^2(\mathrm{Im}\, f) \leq \sum_{\alpha \in J} \mathcal{H}^2(\mathrm{Im}(f|_{f^{-1}(\varphi_\alpha(V_\alpha))})).$$

Since $\alpha \in J$ above was arbitrary, the right hand side of the inequality is zero and $\mathcal{H}^2(\mathrm{Im}\, f) = 0$. □

Metric trees

Before making use of Theorem 5 of Wenger and Young to conclude that contact 3-manifolds have trivial higher Lipschitz homotopy groups, we give a definitions of *metric tree* and *quasi-convexity*.

Definition 6.0.13. A *metric tree*, or $\mathbb{R}$ *tree*, is a metric space (T, d^T) that is uniquely arc-connected in which each arc is isometric to a subarc of $\mathbb{R}$. Given two points x and x' in T, let the geodesic joining x to x' be denoted by $\gamma_{x'}^{x}$ and let the image be denoted by the interval notation $[x, x']$.

The metric d^T on the metric tree T is said to be *convex* since there is equality of values

$$d^T(x, x') = d^T(x, y) + d^T(y, x')$$

for all $x, x' \in T$ and all $y \in [x, x']$. Additionally, the length of the geodesics in T witness this distance between points: for all $x, x' \in T$, the following values agree,

$$l^T(\gamma_{x'}^{x}) = d^T(x, x').$$

Another equivalent characterization of a metric tree is that every (geodesic) triangle is a tripod: For any three points $x, x', x_0 \in T$, there exists a point $x \wedge x'$ such that the intersection of the segments $[x_0, x]$ and $[x_0, x']$ is the segment $[x_0, x \wedge x']$ and the point $x \wedge x'$ is in the segment $[x, x']$.

Observation 6.0.14. Take $x_0 \in T$ to be the base point of the metric tree. For any point $x \in T$, denote the geodesic joining x_0 to x by

$$\gamma_x : [0, d^T(x_0, x)] \hookrightarrow T.$$

From uniqueness of geodesics and every triangle being a tripod, we observe that there is the following equality of restrictions:

$$\gamma_x\big|_{[0, d^T(x_0, x \wedge x')]} = \gamma_{x \wedge x'} = \gamma_{x'}\big|_{[0, d^T(x_0, x \wedge x')]}.$$

Metric trees should be thought of as (simplectic) trees where the extra structure of the metric allows for (unique) geodesics. Much like simplectic trees, metric trees are contractible. To the author's knowledge, the earliest appearance of this result is in Proposition 1.5 of [26]. Further, there is a folklore result that metric trees are Lipschitz contractible. We will now prove this folklore result in the case that the metric tree is compact.

Proposition 6.0.15. *Let (T, d^T) be a compact, metric tree with base point $x_0 \in T$. The metric tree is Lipschitz contractible, that is, there exists a Lipschitz homotopy*

$$H : I \times T \longrightarrow T$$

between the identity map $\mathbb{1}_T$ and the constant map at x_0.

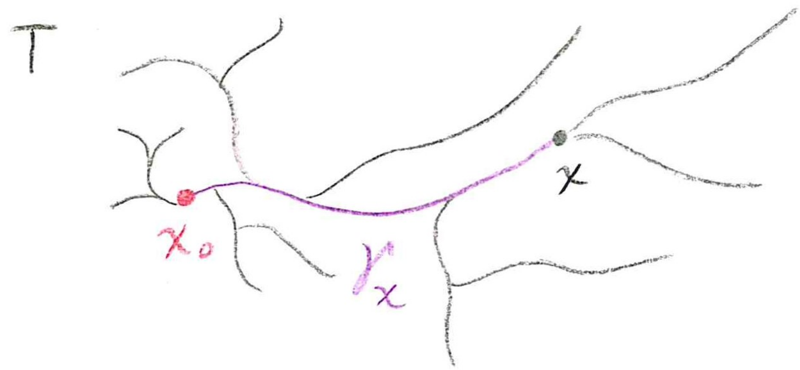

Figure 6.1: The geodesic γ_x joining x_0 and x in the metric tree T.

In order to make sense of a Lipschitz map from the product $I \times T$, we will endow the product space with the metric given by the 1-product metric: for points $(t, x), (t', x') \in I \times T$, the 1-product metric is

$$d^1((t, x), (t', x')) := d^\delta(t, t') + d^T(x, x') = |t - t'| + d^T(x, x'),$$

where d^δ is the standard metric on $\mathbb{R}$.

Proof. We will construct a map that is obviously a homotopy between the identity and the constant and then proceed to show that the map is Lipschitz. We will take advantage of two main properties of metric trees, uniqueness of geodesics and that each geodesic is a isometric embedding of a subinterval of the real numbers.

Before constructing the map, note that, since T is compact, there exists a maximum distance from the base point x_0 to any point x in the metric tree. Denote this value by

$$M := \max_{x \in T} d^T(x_0, x).$$

Define the map H by taking advantage of the geodesic convexity of the metric

tree. Let $H : [0, M] \times T \to T$ be the map defined by the assignment

$$H(t, x) := \begin{cases} x & , t \geq d^T(x_0, x) \\ \gamma_x(t) & , t \leq d^T(x_0, x). \end{cases}$$

For $t = M$, since the value M is the maximum distance of any point in T from the base point, the map $H(M, -) = \mathbb{1}_T$ fixes all points in the metric tree. For $t = 0$, since each geodesic γ_x begins at the base point, the map $H(0, -) = \gamma_{(-)}(0) = x_0$ is constant.

Let $(t, x), (t', x') \in [0, M] \times T$. We proceed by cases to show that the map H is Lipschitz, keying on if the image $H(t', x')$ is contained in the geodesic arc between the base point x_0 and the point x.

Suppose that $H(t', x') \in [x_0, x]$. First, assume that $t' \leq d^T(x_0, x')$. Then, by definition, $H(t', x') = \gamma_{x'}(t')$. Since geodesics are unique, the geodesics γ_x and $\gamma_{x'}$ agree for time t'. Using that geodesics are isometric embeddings of intervals in $\mathbb{R}$, we have the following inequality if $t \leq d^T(x_0, x)$,

$$\begin{aligned} d^T(H(t, x), H(t', x')) &= d^T(\gamma_x(t), \gamma_{x'}(t')) \\ &= d^T(\gamma_x(t), \gamma_x(t')) \\ &= |t - t'| \\ &\leq d^1((t, x), (t', x')). \end{aligned}$$

If $t \geq d^T(x_0, x)$, since $H(t, -)$ fixes the point x, it is described as the end point

of the geodesic γ_x. Thus, we have the following inequality:

$$\begin{aligned} d^T(H(t,x),H(t',x')) &= d^T(x,\gamma_{x'}(t')) \\ &= d^T(\gamma_x(d^T(x_0,x)),\gamma_x(t')) \\ &= |d^T(x_0,x)-t'| \\ &\leq |t-t'| \\ &\leq d^1((t,x),(t',x')). \end{aligned}$$

Now, suppose that $t' \geq d^T(x_0,x')$. Then, the map $H(t',-)$ fixes the point x'. Thus, we can express the the image of (t',x') as a point on the geodesic $H(t',x') = x' = \gamma_x(d^T(x_0,x'))$.

We now key on the value t and how it compares to distances $d^T(x_0,x')$ and $d^T(x_0,x)$. If $t \geq d^T(x_0,x)$, we can use convexity of the metric d^T and that the geodesic γ_x is an isometric embedding to establish the following inequality:

$$\begin{aligned} d^T(H(t,x),H(t',x')) &= d^T(\gamma_x(d^T(x_0,x)),\gamma_x(d^T(x_0,x'))) \\ &= |d^T(x_0,x)-d^T(x_0,x')| \\ &\leq d^T(x,x') \\ &\leq d^1((t,x),(t',x')). \end{aligned}$$

If $d^T(x_0, x') \leq t \leq d^T(x_0, x)$, we can similarly use convexity:

$$\begin{aligned} d^T(H(t,x), H(t',x')) &= d^T(\gamma_x(t), \gamma_x(d^T(x_0, x'))) \\ &= |t - d^T(x_0, x')| \\ &\leq |d^T(x_0, x) - d^T(x_0, x')| \\ &\leq d^T(x, x') \\ &\leq d^1((t,x),(t',x')). \end{aligned}$$

Finally, if $t \leq d^T(x_0, x)$ and $t \leq d^T(x_0, x')$, we use that $t' \geq d^T(x_0, x')$:

$$\begin{aligned} d^T(H(t,x), H(t',x')) &= d^T(\gamma_x(t), \gamma_x(d^T(x_0, x'))) \\ &= |d^T(x_0, x') - t| \\ &\leq |t' - t| \\ &\leq d^1((t,x),(t',x')). \end{aligned}$$

<u>Suppose that $H(t', x') \notin [x_0, x]$.</u> Keying in on the geodesic arc connecting $H(t', x')$ and $H(t, x)$, there are four distinct cases depending on how t and t' compare to $d^T(x_0, x)$ and $d^T(x_0, x')$, respectively. For each of those four cases, the geodesic arc connecting $H(t', x')$ to $H(t, x)$ can be described as

$$[H(t', x'), H(t, x)] = [\gamma_{x'}(t'_0), \gamma_x(t_0)]$$

for some value $t'_0 \in [d^T(x_0, x \wedge x'), d^T(x_0, x')]$ and some value $t_0 \in [d^T(x_0, x \wedge x'), d^T(x_0, x)]$. In any of these cases, the geodesic arc is contained in the geodesic arc

$[x', x]$. Thus, the distance from x' to x bounds the distance between the images:

$$\begin{aligned} d^T(H(t,x), H(t',x')) &\leq d^T(x,x') \\ &\leq d^1((t,x),(t',x')). \end{aligned}$$

Swapping the roles of $t', x')$ and (t, x) yields identical arguments that complete the cases argument. In each case, we have shown the inequality $d^T(H(t,x), H(t',x')) \leq d^1((t,x),(t',x'))$. Thus, the map H is Lipschitz. Therefore, the compact metric tree T is Lipschitz contractible.

□

We now consider subtrees of a metric tree. We will show that, for any metric tree and any subtree, there is a Lipschitz deformation retract from onto the subtree.

Definition 6.0.16. Let (T, d^T) be a metric tree. A *subtree* $S \subset T$ is a compact, connected, nonempty subset of the metric tree T.

Observation 6.0.17. A subtree S in a metric tree (T, d^T) is a metric tree with the restricted metric $d^T|_S$. Indeed, since T is uniquely arc-connected and S is connected, the unique arc is contained in S. So, the subtree S is uniquely arc-connected. Moreover, between any two points in T, the unique arc is a geodesic witnessing the distance between its endpoints. Thus, the restricted metric $d^T|_S$ evaluated on any two points agrees with the length of the geodesic contained in S connecting the endpoints.

Lemma 6.0.18. *Let (T, d^T) be a metric tree and let $S \subset T$ be a subtree of T. For any point $x \in T$, there exists a unique point $p(x) \in S$ that witnesses the distance from x to the subtree: $d^T(x, S) = d^T(x, p(x))$. Moreover, on any connected component of $T \smallsetminus S$, the map $p : T \to S$ is constant.*

Proof. Let $x \in T$ be a point in the metric tree T. The real-valued function $d^T(x, -) : S \to \mathbb{R}$ is continuous on the compact set S. Thus, there exists a point $p(x) \in S$ such that the function achieves its minimum. Since distance from a set is defined as an infimum, $d^T(x, S) = d^T(x, p(x))$.

If the point x is in the subtree S, the point $p(x)$ is the point x right back. By positive definiteness of the metric, this is the unique closest point in S to x.

Let $x \in T \smallsetminus S$. Suppose there is a point $p(x)' \in S$ such that $d^T(x, p(x)') = d^T(x, S) = d^T(x, p(x))$. Let $[x, p(x)']$ be the unique arc joining x and $p(x)'$ and let $[x, p(x)]$ be the unique arc joining x and $p(x)$.

Consider the intersection of the arcs $[x, p(x)']$ and $[x, p(x)]$. Then, there is a unique point $p(x) \wedge p(x)'$ such that the intersection is the arc

$$[x, p(x)'] \cap [x, p(x)] = [x, p(x) \wedge p(x)'].$$

If the point $p(x) \wedge p(x)'$ is in $T \smallsetminus S$, then the unique arc connecting $p(x)$ and $p(x)'$ contains a point in the complement of S, namely $p(x) \wedge p(x)'$. This contradicts that S is uniquely arc-connected. Suppose that the point $p(x) \wedge p(x)'$ is in S. Since the arc $[x, p(x) \wedge p(x)']$ is contained in $[x, p(x)]$, we have an inequality of the associated distances

$$d^T(x, p(x) \wedge p(x)') \leq d^T(x, p(x))$$

and similarly, $d^T(x, p(x) \wedge p(x)') \leq d^T(x, p(x)')$. These inequalities are equalities if and only if $p(x) \wedge p(x)' = p(x)$ and $p(x) \wedge p(x)' = p(x)'$. So, the points $p(x)$ and $p(x)'$ are not distinct. Thus, $p(x) = p(x)'$ is the unique closest point to x in S.

Now, we will show that the selection of a point in S by the map p is determined by the connected component of $T \smallsetminus S$. Let $x, x' \in T \smallsetminus S$ be in the same connected component $A \subset T \smallsetminus S$.

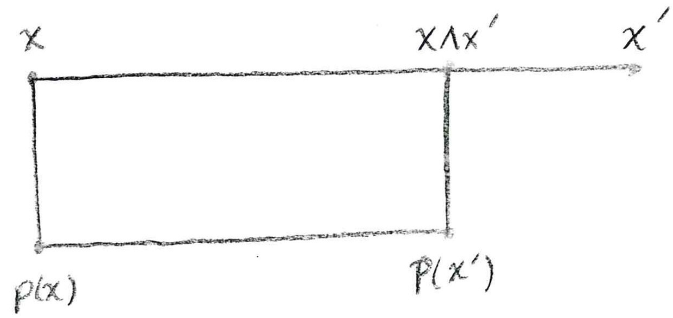

Figure 6.2: If $p(x)$ and $p(x')$ were distinct points. They will be shown to be equal.

Since the component of $T \setminus S$ is connected, the set A contains the arc $[x, x']$ connecting the points x and x'. Since $p(x)$ is the closest point in S to x, the intersection of the arc $[x, p(x)]$ and the subtree S is the singleton set containing $p(x)$. Likewise, $[x', p(x')] \cap S = \{p(x')\}$.

By the tripod condition of metric trees, there exists a point $x \wedge x'$ in T such that there is an equality of sets

$$[x, p(x')] \cap [x', p(x')] = [x \wedge x', p(x')]$$

and $x \wedge x' \in [x, x']$. Since the component A contains the arc $[x, x']$, it also contains the sub arc $[x, x \wedge x'] \subset A$. Also, since the sub arc $[x \wedge x', p(x')]$ is contained in the arc $[x', p(x')]$, the intersection of the sub arc with the subtree S is the singleton set containing $p(x')$,

$$[x \wedge x', p(x')] \cap S = \{p(x')\}.$$

By the uniqueness of arcs connecting points in a metric tree, the arc $[x, p(x')]$ can be described as the union of arcs

$$[x, p(x')] = [x, x \wedge x'] \bigcup [x \wedge x', p(x')].$$

So, the arc $[x, p(x')]$ intersects the subtree S at one point $p(x')$. Also, $[x, p(x')] \setminus \{p(x')\}$ is contained in $A \subset T \setminus S$.

Now, the arc $[p(x), p(x')]$ is contained in the subtree S. Thus, taking the union of this arc and the arc $[x, p(x)]$ yields another arc. In fact, by uniqueness, the result of the union is the arc $[x, p(x')]$. Thus, we have the following equality

$$[x, x \wedge x'] \bigcup [x \wedge x', p(x')] = [x, p(x')] = [x, p(x)] \bigcup [p(x), p(x')].$$

Since the arc $[x, p(x')]$ intersects the subtree S at exactly the point $p(x')$ and the arc $[p(x), p(x')]$ is contained in S, the arc $[p(x), p(x')]$ is a singleton set with the element $p(x') = p(x)$. Thus, the assignment of closest point in S depends on the component $A \subset T \setminus S$.

□

Lemma 6.0.19. *Let (T, d^T) be a compact metric tree and let $S \subset T$ be a subtree. There exists a Lipschitz deformation retract from T onto S.*

Proof. We will produce a Lipschitz homotopy by contracting each component of $T \setminus S$ onto the unique point of S that is closest to the component. We will fix S through this homotopy.

Let $\{A_\alpha\}_{\alpha \in J}$ be the collection of connected components of the subspace $T \setminus S$, where J is come indexing set. For each $\alpha \in J$, Lemma 6.0.18 guarantees that there is a unique point $p(A_\alpha)$ in S such that, for all points $x \in A_\alpha$, the closest point to x in S is $p(A_\alpha)$.

$$d^T(x, S) = d^T(x, p(A_\alpha)).$$

Consider the closure of the component $\overline{A_\alpha}$. The set is closed and thus compact. The set $\overline{A_\alpha}$ contains the point $p(A_\alpha)$ since, for any point $x \in A_\alpha$, the set $[p(A_\alpha), x] \setminus$

$\{p(A_\alpha)\}$ is contained in A_α and the closure is the arc containing the boundary point $p(A_\alpha)$. In fact, $\overline{A_\alpha} = A_\alpha \cup \{p(A_\alpha)\}$ as any point $y \in T \smallsetminus (A_\alpha \cup \{p(A_\alpha)\})$ has an open set $[y, p(A_\alpha)] \smallsetminus \{p(A_\alpha)\}$ that does not intersect the component A_α.

The compact set $\overline{A_\alpha}$ is also connected. Thus, $\overline{A_\alpha}$ is a subtree in T. From Observation 6.0.17, $\overline{A_\alpha}$ is a metric tree. Consider the point $p(A_\alpha)$ as the base point. By Proposition 6.0.15, there exists a Lipschitz map

$$H_\alpha : I \times \overline{A_\alpha} \longrightarrow \overline{A_\alpha}$$

between the identity map on $\overline{A_\alpha}$ and the constant map $p(A_\alpha)$.

Define the map $F : I \times T \longrightarrow T$ by the assignment

$$F(t,x) := \begin{cases} H_\alpha(t,x) & , x \in \overline{A_\alpha} \\ x & , x \in S. \end{cases}$$

If $x \in S$, then $H_\alpha(t,x) = x$ for all time t by construction of H_α. Thus, $F|_{I \times S} = \mathbb{1}_S$. Also, for any point $x \in A_\alpha$ and $t = 1$, since H_α is a deformation retract onto $p(A_\alpha)$,

$$F(1,x) = H_\alpha(1,x) = p(A_\alpha) \in S.$$

So, F is a deformation retract from T onto the subtree S. The homotopy F is Lipschitz since it is constructed from Lipschitz maps.

$\square$

Definition 6.0.20. A metric space (X,d) is *quasi-convex* if there exists positive value C such that, for any two points $x, x' \in X$, there exists a path γ joining the points

such that the length of the path is at most $C \cdot d(x, x')$:

$$l(\gamma) \leq C \cdot d(x, x').$$

<u>Higher Lipschitz homotopy groups of contact 3-manifolds are trivial</u>

Corollary 6.0.21. *If $n \geq 2$ and $\alpha : \mathbb{S}^n \longrightarrow M$ is a Lipschitz map into a contact 3-manifold (M, ξ), then α is Lipschitz null-homotopic. That is, $\pi_n^{Lip}(M, \xi) = 0$.*

Proof. This follows directly from Theorem 5 in [34]. $\mathbb{S}^n$, with its standard Riemannian metric, is quasi-convex. $\mathbb{S}^n$ is also simply connected and it follows that $\pi_1^{\mathrm{Lip}}(\mathbb{S}^n) = 0$ (See Example 3.0.66). By Theorem 6.0.12, (M, ξ) is purely 2-unrectifiable. Thus, by Theorem 5 in [34], α factors through a metric tree Z,

$$\begin{tikzcd} \mathbb{S}^n \arrow[rr, "\alpha"] \arrow[dr, "\psi"'] & & M \\ & Z. \arrow[ur, "\phi"'] & \end{tikzcd}$$

The maps $\psi : \mathbb{S}^n \longrightarrow Z$ and $\phi : Z \longrightarrow M$ are Lipschitz as well.

Since Z is a metric tree, by Proposition 6.0.15, it is contractible by a Lipschitz homotopy $h : Z \times [0, 1] \longrightarrow Z$. Therefore, the homotopy $H : \mathbb{S}^n \times [0, 1] \longrightarrow M$ given by $H(p, t) := \phi(h(\psi(p), t))$ is a Lipschitz null-homotopy of α.

$\square$

FIRST LIPSCHITZ HOMOTOPY GROUPS OF CONTACT 3-MANIFOLDS

The main focus of this chapter is identifying properties of $\pi_1^{\mathrm{Lip}}(M, d_{CC}^M)$ for an arbitrary contact 3-manifold, endowed with a compatible sub-Riemannian structure. As will be shown, this group is uncountably generated no matter the choice of contact 3-manifold or sub-Riemannian structure. The heart of this result is that (M, d_{CC}^M) is purely 2-unrectifiable. As such, the image of any Lipschitz map, in particular any Lipschitz homotopy, in (M, d_{CC}^M) will not have positive $\mathcal{H}^2$-measure. Then, for distinct Lipschitz loops, no Lipschitz homotopy will be able to sweep out the "area" bounded by the curves. Thus, two Lipschitz loops being Lipschitz homotopic will be shown to be unlikely.

The argument that $\pi_1^{\mathrm{Lip}}(M, d_{CC}^M)$ is uncountably generated for an arbitrary contact 3-manifold will go as follows: First, a minor result about integration with respect to a measure will be recorded. We will then proceed by again focusing on horizontal embeddings of $\mathbb{S}^1$ into the contact 3-manifold (M, ξ). It is known that such horizontal maps are Lipschitz with respect to the Carnot-Carathéodory metric (Theorem 3.1 in [6]). It will then be shown that the image of a Lipschitz homotopy between such embeddings would have positive $\mathcal{H}^2$-measure. This will be shown, in part, by using the 1-form ω constructed earlier in Lemma 4.0.11 and using a Lipschitz version of Stokes' theorem. Since (M, d_{CC}^M) is purely 2-unrectifiable, the image of any Lipschitz homotopy into (M, d_{CC}^M) cannot have positive $\mathcal{H}^2$-measure and, thus, there will be no Lipschitz homotopies between distinct horizontal embeddings of $\mathbb{S}^1$.

Therefore, each distinct horizontal embedding of $\mathbb{S}^1$ into (M, ξ) will yield a distinct element in the group $\pi_1^{\mathrm{Lip}}(M, d_{CC}^M)$. As there are uncountably many horizontal embeddings of $\mathbb{S}^1$ into (M, ξ) (Lemma 4.0.22), the group $\pi_1^{\mathrm{Lip}}(M, d_{CC}^M)$ will be shown to be uncountable and, in fact, uncountably generated. This last result will follow

nearly identically to the $\pi_1^{\mathsf{H}}(M,\xi)$ case.

Throughout this chapter, let

$$\gamma_1, \gamma_2 : \mathbb{S}^1 \hookrightarrow (M,\xi)$$

be horizontal embeddings into some contact 3-manifold (M,ξ) endowed with a sub-Riemannian structure such that the images of the embeddings do not agree. In the smooth case, it was shown that a smooth homotopy between γ_1 and γ_2 must have rank 2 somewhere (Lemma 4.0.15). In the Lipschitz case, it will be shown that a Lipschitz homotopy between γ_1 and γ_2 must have image with positive Hausdorff 2-measure. Both of these results capture a sense in which a homotopy between distinct embeddings of $\mathbb{S}^1$ needs to sweep out area in order to transition from one knot to the other.

In order to guarantee that the image of such a Lipschitz homotopy H will have positive $\mathcal{H}^2$-measure, we will find a measurable function that yields a non-zero integral when integrated against $\mathcal{H}^2$ over the image $\mathrm{Im}(H)$. This is enough to guarantee that the 2-dimensional Hausdorff measure of the image of the homotopy H is positive, $\mathcal{H}^2(\mathrm{Im}(H)) > 0$.

Before proceeding, we make note of a standard result in measure theory. See for example [3] or [10] for the basics of measure theory. We will use this result in forthcoming arguments as a tool to show that certain measure spaces do not have measure zero.

Lemma 7.0.1. *Let (X,μ) be a measure space and $f : X \longrightarrow [-\infty,\infty]$ be an integrable function on X. Then, if $\mu(X) = 0$, then the integral $\int_X f \ d\mu = 0$ vanishes. Thus, if there exists an integrable function f such that the integral $\int_X f \ d\mu \neq 0$ is non-zero, then the set X has positive measure.*

Lipschitz Homotopies Between Distinct Embeddings of $\mathbb{S}^1$

Now, it will be shown that there is a result parallel to Theorem 4.0.28 in the case where horizontal maps are replaced with Lipschitz maps with respect to a sub-Riemannian structure on a contact 3-manifold. The result will state that the first Lipschitz homotopy group of a contact 3-manifold is uncountably generated, just as Theorem 4.0.28 in the horizontal case.

The argument is extremely similar to the horizontal case, the major differences being that we will work with Lipschitz maps primarily into $\mathbb{R}^k$ and Stokes' theorem will be replaced by a Lipschitz version found in [6]. For convenience, the statement of of this Lipschitz Stokes' theorem is presented currently:

Lemma 7.0.2 ([6] Lemma 4.9.). *If $g : \mathbb{D}^{n+1} \to \mathbb{R}^k$, $k \geq n$, is Lipschitz and ω is a smooth n-form on $\mathbb{R}^k$, then*

$$\int_{\partial \mathbb{D}^{n+1}} g^* \omega = \int_{\mathbb{D}^{n+1}} g^* (d\omega).$$

Convention 7.0.3. Here and going forward, $\mathbb{R}^k$, and any subset thereof, is endowed with the Lebesgue measure $\mathcal{L}^n$ and the Euclidean metric unless otherwise mentioned.

Pullbacks of smooth forms by Lipschitz maps

Something of note in this Lipschitz Stokes' theorem is that these forms on $\mathbb{R}^k$ are being pulled back to forms on $\mathbb{D}^{n+1}$ via a Lipschitz map, not a smooth map, which is the standard means of pulling back a form. The pullbacks $g^*\omega$ and $g^*(d\omega)$ are still defined almost everywhere though as, via Rademacher's Theorem [10], the pushforward map g_* is defined everywhere but on a set of measure zero.

Theorem 7.0.4. *(Rademacher's Theorem, Theorem 3.2 in [10]) Let $g : \mathbb{R}^n \to \mathbb{R}^k$ be*

a locally Lipschitz function. Then, g is differentiable $\mathcal{L}^n$-almost everywhere.

So, take any locally Lipschitz function $g : \mathbb{R}^{n+1} \to \mathbb{R}^k$ and any smooth k-form ω on $\mathbb{R}^k$. Then, for almost every point $p \in \mathbb{R}^{n+1}$ and any collection of k tangent vectors $X_1, \ldots, X_k \in T_p\mathbb{R}^{n+1}$, the pullback of the form ω is defined:

$$(g^*\omega)_p(X_1, \ldots, X_k) = \omega_{g(p)}(D_p g X_1, \ldots, D_p g X_k).$$

Since the pullback $g^*\omega$ is defined everywhere on $\mathbb{R}^{n+1}$ outside of a $\mathcal{L}^n$-measure zero set, the integral of the pullback is defined.

With the notion of pullback of a smooth form via a Lipschitz map established, it will be argued that a Lipschitz homotopy between distinct smooth embeddings of $\mathbb{S}^1$ into $\mathbb{R}^k$ must have positive $\mathcal{H}^2$-measure. First, a similar result to Lemma 4.0.12 is shown in the case that locally Lipschitz maps are considered rather than smooth maps.

Lemma 7.0.5. *Let $n \geq 2$. Let $B \subset \mathbb{R}^n$ be a compact and measurable set, let $H : B \longrightarrow \mathbb{R}^k$ be a locally Lipschitz map with respect to the standard Euclidean metrics, and let $\theta \in \Omega^n(\mathbb{R}^k)$ be a smooth n-form on $\mathbb{R}^k$. If $\int_B H^*\theta \neq 0$, then H is rank n on a set of positive measure.*

Proof. First, it will be shown that there is a set of positive measure on which $H^*\theta$ is not identically zero. Then, it will be argued that H must have rank n everywhere that $H^*\theta \not\equiv 0$. Thus, H will have rank n on a set of positive measure.

Denote by B' the set of all points of the set B where the map H is differentiable. The set B' is measurable since B is assumed to be measurable and the set of points where the map H is not differentiable is measurable.

Proceed by naming the subset of B' on which $H^*\theta$ is not identically zero. Define the subset $A \subset B'$ as

$$A := \{p \in B' \ : \ (H^*\theta)_p \not\equiv 0\}.$$

The set A will be shown to be a measurable subset of B'.

The n-form $H^*\theta$ determines a continuous map from B' to the nth exterior power of the cotangent bundle of $\mathbb{R}^k$:

$$B' \xrightarrow{H^*\theta} \Lambda^n(T^*\mathbb{R}^k).$$

Fix a vector bundle isomorphism

$$\begin{tikzcd} \Lambda^n(T^*\mathbb{R}^k) \arrow[rr, "\psi", "\cong"'] \arrow[dr] & & \mathbb{R}^k \times \mathbb{R}^{\binom{k}{n}} \arrow[dl] \\ & \mathbb{R}^k, & \end{tikzcd}$$

and consider the composition of continuous maps,

$$B' \xrightarrow{H^*\theta} \Lambda^n(T^*\mathbb{R}^k) \xrightarrow{\psi} \mathbb{R}^k \times \mathbb{R}^{\binom{k}{n}}.$$

The subset $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^k \times \mathbb{R}^{\binom{k}{n}}$ is measurable and thus so is its preimage with respect to this continuous composition of maps:

$$(\psi \circ H^*\theta)^{-1}\left(\mathbb{R}^k \times \{0\}\right) = (H^*\theta)^{-1}\left(\psi^{-1}\left(\mathbb{R}^k \times \{0\}\right)\right) = \{p \in B' \ : \ (H^*\theta)_p \equiv 0\}.$$

This last equality follows from ψ being a vector bundle isomorphism. It follows that the set $\{p \in B' \ : \ (H^*\theta)_p \equiv 0\}$ is measurable, as is the set

$$A = \{p \in B' \ : \ (H^*\theta)_p \not\equiv 0\} = B' \smallsetminus \{p \in B' \ : \ (H^*\theta)_p \equiv 0\}.$$

We will show that A is a set of positive measure. Assume by way of contradiction that the set A is measure zero. Then the integral $\int_A H^*\theta$ vanishes by Lemma 7.0.1. Also, the form $H^*\theta$ is identically zero on $B' \setminus A$, so the integral $\int_{B'\setminus A} H^*\theta$ vanishes as well. Thus, the integral

$$\int_{B'} H^*\theta = \int_A H^*\theta + \int_{B'\setminus A} H^*\theta = 0,$$

vanishes, contradicting the assumption that this integral is non-zero. So, the set A has positive measure.

Note that the map H is differentiable on the set A. It will be shown that H must have rank n on the entirety of A. Proceeding by contradiction, suppose that the rank of H was strictly less than n at some point $p \in A$. As will be shown, this implies that the form $(H^*\theta)_p$ is identically zero.

Let $v_1, \ldots, v_n \in T_pN$. Since the rank of D_pH is strictly less than n, the image of these vectors under the map D_pH cannot span $T_{H(p)}M$. So, there exists a vector in the above list v_j such that

$$D_pHv_j = \sum_{\substack{i=1 \\ i\neq j}}^{n} c_i \cdot D_pHv_i$$

for scalars $c_i \in \mathbb{R}$. Thus,

$$\begin{aligned}
(H^*\theta)_p(v_1, \ldots, v_j, \ldots, v_n) &= \theta_{H(p)}(D_pHv_1, \ldots, D_pHv_j, \ldots, D_pHv_n) \\
&= \theta_{H(p)}(D_pHv_1, \ldots, \sum_{\substack{i=1 \\ i\neq j}}^{n} c_i \cdot D_pHv_i, \ldots, D_pHv_n) \\
&= \sum_{\substack{i=1 \\ i\neq j}}^{n} c_i \cdot \theta_{H(p)}(D_pHv_1, \ldots, D_pHv_i, \ldots, D_pHv_i, \ldots, D_pHv_n) \\
&= 0.
\end{aligned}$$

This last equality follows from $\theta_{H(p)}$ being an alternating n-tensor and the vector

D_pHv_i appearing twice as input in each term of the summation.

As the vectors $v_1, \ldots, v_n$ were arbitrary, it follows that $(H^*\theta)_p \equiv 0$. This contradicts that p is an element in A. Thus, H has rank n on A, a set of positive measure.

□

Lipschitz homotopies sweep out area

It is now shown that a Lipschitz homotopy H between smooth embeddings of $\mathbb{S}^1$ into $\mathbb{R}^k$ must sweep out "area," i.e., $\mathcal{H}^2(\mathrm{Im}(H)) > 0$.

In the following lemma, assume $[1,2] \times \mathbb{S}^1$ lives inside $\mathbb{R}^2$ via polar coordinates as the standard annulus centered at the origin of inner radius 1 and outer radius 2. $\{1\} \times \mathbb{S}^1$ is associated to the unit circle and $\{2\} \times \mathbb{S}^1$ is associated to the circle of radius 2 centered at the origin. Endow $[1,2] \times \mathbb{S}^1 \subset \mathbb{R}^2$ with a metric by restricting the Euclidean metric on $\mathbb{R}^2$.

Lemma 7.0.6. *Let $\gamma_1, \gamma_2 : \mathbb{S}^1 \hookrightarrow \mathbb{R}^k$, $k \geq 2$ be smooth embeddings of $\mathbb{S}^1$ such that their images are distinct: $\mathrm{Im}(\gamma_1) \neq \mathrm{Im}(\gamma_2)$. Let $H : [1,2] \times \mathbb{S}^1 \longrightarrow \mathbb{R}^k$ be a Lipschitz homotopy between γ_1 and γ_2. Then, the 2-dimensional Hausdorff measure of the image of H is positive:*

$$\mathcal{H}^2(\mathrm{Im}(H)) > 0.$$

Proof. As in the the proof of Lemma 4.0.15, we will produce a 2-form on $\mathbb{R}^k$ such that the integral of the pullback via the Lipschitz map H is nonzero. This will guarantee that H has rank 2 on a set of positive measure via Lemma 4.0.12. In turn, this will be enough to yield that $\mathcal{H}^2(\mathrm{Im}(H)) > 0$.

Take the 1-form $\omega \in \Omega^1(\mathbb{R}^k)$ guaranteed by Lemma 4.0.11 with respect to embeddings γ_1 and γ_2. We will consider the 2-form $d\omega$ on $\mathbb{R}^k$.

Next, it will be shown that $\int_{[1,2]\times\mathbb{S}^1} H^* d\omega$ is nonzero. Lemma 7.0.2 will be of significant importance to the calculation. As this result only applies to maps with domain isomorphic to $\mathbb{D}^2$, it is necessary that the Lipschitz map H be extended to the closed ball of radius 2 centered at the origin in $\mathbb{R}^2$, denoted $\mathbb{D}^2(2)$. Via Theorem 3.1 in [10], a Lipschitz map defined on a subset of Euclidean space can be extended to a Lipschitz map on the entire ambient space. Thus, the Lipschitz map H can be extended to a Lipschitz map defined on all of $\mathbb{R}^2$:

$$\tilde{H} : \mathbb{R}^2 \longrightarrow \mathbb{R}^k,$$

where $\tilde{H}|_{[1,2]\times\mathbb{S}^1} = H$.

As there is an equality of sets $\mathbb{D}^2(2) = ([1,2] \times \mathbb{S}^1) \cup \mathbb{D}^2$ such that the overlap of its indicated subsets is $\{1\} \times \mathbb{S}^1$, there is an equality of integrals,

$$\int_{\mathbb{D}^2(2)} \tilde{H}^* d\omega = \int_{[1,2]\times\mathbb{S}^1} \tilde{H}^* d\omega + \int_{\mathbb{D}^2} \tilde{H}^* d\omega.$$

Lemma 7.0.2 will be used to show that $\int_{\mathbb{D}^2(2)} \tilde{H}^*(d\omega) \neq 0$ and $\int_{\mathbb{D}^2} \tilde{H}^*(d\omega) = 0$. Once these properties are established, since $\tilde{H}$ is an extension of H, it will be immediate that the integral

$$\int_{[1,2]\times\mathbb{S}^1} H^* d\omega = \int_{[1,2]\times\mathbb{S}^1} \tilde{H}^* d\omega = \int_{\mathbb{D}^2(2)} \tilde{H}^* d\omega - \int_{\mathbb{D}^2} \tilde{H}^* d\omega \neq 0$$

is non-zero.

First, consider the integral $\int_{\mathbb{D}^2} \tilde{H}^* d\omega$. Via Lemma 7.0.2, we have an equality of integrals

$$\int_{\mathbb{D}^2} \tilde{H}^* d\omega = \int_{\{1\}\times\mathbb{S}^1} \tilde{H}^* \omega,$$

as $\partial\mathbb{D}^2 = \{1\}\times\mathbb{S}^1$. Since the restriction of the extension $\tilde{H}|_{\{1\}\times\mathbb{S}^1} = H|_{\{1\}\times\mathbb{S}^1} = \gamma_1$ agrees with the embedding γ_1, by the construction of ω in Lemma 4.0.11, this integral vanishes,

$$\int_{\{1\}\times\mathbb{S}^1} \tilde{H}^*\omega = \int_{\{1\}\times\mathbb{S}^1} \gamma_1^*\omega = 0.$$

Now, consider $\int_{\mathbb{D}^2(2)} \tilde{H}^* d\omega$. Again, via Lemma 7.0.2, we have an equality of integrals,

$$\int_{\mathbb{D}^2(2)} \tilde{H}^* d\omega = \int_{\mathbb{S}^1_2} \tilde{H}^*\omega,$$

as $\partial\mathbb{D}^2(2) = \{2\}\times\mathbb{S}^1$. Since the restriction of the extension $\tilde{H}|_{\{2\}\times\mathbb{S}^1} = H|_{\{2\}\times\mathbb{S}^1} = \gamma_2$ agrees with the embedding γ_2, by construction of ω, this integral is non-zero,

$$\int_{\{2\}\times\mathbb{S}^1} \tilde{H}^*\omega = \int_{\{2\}\times\mathbb{S}^1} \gamma_2^*\omega \neq 0.$$

Therefore, we have shown that the integral

$$\int_{[1,2]\times\mathbb{S}^1} H^* d\omega = \int_{\{2\}\times\mathbb{S}^1} \gamma_2^*\omega - \int_{\{1\}\times\mathbb{S}^1} \gamma_1^*\omega \neq 0$$

is non-zero. By Lemma 7.0.5, H has rank 2 on a set of positive measure.

The proof will be completed by verifying that H having rank 2 on a set of positive measure implies that the Hausdorff measure of the image of H is positive.

Let $p \in [1,2]\times\mathbb{S}^1$ be a point at which the linear map

$$D_pH : T_p([1,2]\times\mathbb{S}^1) \to T_{H(p)}\mathbb{R}^k$$

has rank 2. Fixing coordinates on $\mathbb{R}^2$ and $\mathbb{R}^k$ allows for D_pH to be understood at a $(k\times 2)$-matrix. Since D_pH is rank 2, the (2×2)-matrix $(D_pH)^T(D_pH)$ is invertible

and thus has positive determinant:

$$J_H(p) := \sqrt{\det((D_pH)^T(D_pH))} > 0.$$

As p was arbitrary, the *Jacobian map* $J_H : [1,2] \times \mathbb{S}^1 \to \mathbb{R}$ is positive on a set of positive measure. Thus, its integral is positive,

$$\int_{[1,2]\times\mathbb{S}^1} J_H \; dA > 0,$$

where dA is an area form on $\mathbb{R}^2$.

By the Area Formula (Theorem 3.8 in [10]), we have an equality of integrals

$$\int_{\mathrm{Im}(H)} \mathcal{H}^0(H^{-1}(y)) \; d\mathcal{H}^2(y) = \int_{[1,2]\times\mathbb{S}^1} J_H \; dA > 0,$$

where $\mathcal{H}^0$ is the counting measure. Via Lemma 7.0.1, since a measurable function integrates against the measure $\mathcal{H}^2$ to a non-negative value, the 2-dimensional Hausdorff measure of the image of H is positive: $\mathcal{H}^2(\mathrm{Im}(H)) > 0$.

□

<u>$\pi_1^{\mathrm{Lip}}(M, d_{CC}^M)$ is uncountably generated</u>

Now that it has been verified that Lipschitz homotopies between distinct embeddings of $\mathbb{S}^1$ into $\mathbb{R}^k$ must sweep out a region of positive $\mathcal{H}^2$-measure, we will use that (M, d_{CC}^M) is purely 2-unrectifiable to show that no such Lipschitz homotopies exist in contact 3-manifolds.

Distinct horizontal embeddings of $\mathbb{S}^1$ are not Lipschitz homotopic

Corollary 7.0.7. *Let $\gamma_1, \gamma_2 : \mathbb{S}^1 \hookrightarrow (M, \xi)$ be horizontal embeddings of $\mathbb{S}^1$ into a 3-dimensional contact manifold (M, ξ) such that $\mathrm{Im}(\gamma_1) \neq \mathrm{Im}(\gamma_2)$. Then γ_1 and γ_2 are not Lipschitz homotopic.*

Proof. Assume that such a Lipschitz homotopy $H : [1,2] \times \mathbb{S}^1 \to (M, d_{CC}^M)$ exists between γ_1 and γ_2. It will be shown that H yields a Lipschitz map from an annulus in $\mathbb{R}^2$ to some high dimensional $\mathbb{R}^k$. Since the contact 3-manifold (M, d_{CC}^M) is purely 2-unrectifiable, this map into $\mathbb{R}^k$ will have Hausdorff 2-measure 0. This will then contradict Lemma 7.0.6 and the proof will be complete.

First, the Lipschitz map H is used to find a Lipschitz map from an annulus into $\mathbb{R}^k$. To begin, the Lipschitz map H is used to find a Lipschitz map into a Riemannian manifold. Let $\tilde{g}$ be any Riemannian metric on M. By Theorem 2.10 of [24], the identity map from the sub-Riemannian manifold (M, d_{CC}^M) to the Riemannian manifold $(M, \tilde{g})$ is locally Lipschitz. Thus, the map

$$\mathbb{1}_M \circ H : [1,2] \times \mathbb{S}^1 \longrightarrow (M, d^{\tilde{g}})$$

is a locally Lipschitz map into M with the path metric $d^{\tilde{g}}$ associated to the Riemannian metric $\tilde{g}$.

By Whitney embedding theorem, there exists $k \geq 3$ such that there is a smooth embedding $\iota : M \hookrightarrow \mathbb{R}^k$ of the manifold M into k-dimensional Euclidean space. Endow $\mathbb{R}^k$ with its standard Euclidean metric d^δ. By Lemma 2.0.60, the embedding ι is a locally Lipschitz map with respect to the associated metrics. Thus, the composite of locally Lipschitz maps

$$[1,2] \times \mathbb{S}^1 \xrightarrow{H} (M, d_{CC}^M) \xrightarrow{\mathbb{1}_M} (M, d^{\tilde{g}}) \overset{\iota}{\hookrightarrow} (\mathbb{R}^k, d^\delta)$$

is a locally Lipschitz map. Since this map has compact domain, the composite is in fact a Lipschitz map (Lemma 2.0.63).

Proceeding, it will be argued that the image of this composite map has Hausdorff 2-measure zero with respect to the Euclidean metric on $\mathbb{R}^k$. This will yield a contradiction as Lemma 7.0.6 states that a Lipschitz homotopy between smooth embeddings of $\mathbb{S}^1$ must have positive Hausdorff 2-measure.

First, since H is a Lipschitz map with domain contained in $\mathbb{R}^2$ and (M, d_{CC}^M) is purely 2-unrectifiable (Theorem 6.0.12), the Hausdorff 2-measure of the image of H in M with respect to the Carnot-Carathéodory metric is 0: $\mathcal{H}^2_{CC}(\mathrm{Im}(H)) = 0$.

Now, as cited above, the identity map on M,

$$\mathbb{1}_M : (M, d_{CC}^M) \longrightarrow (M, d^{\tilde{g}}),$$

is locally Lipschitz. By Lemma 2.0.69, the Hausdorff 2-measure of the image $\mathrm{Im}(\mathbb{1}_M \circ H) \subset M$ with respect to the metric $d^{\tilde{g}}$ is 0: $\mathcal{H}^2_{\tilde{g}}(\mathrm{Im}(\mathbb{1}_M \circ H)) = 0$. Finally, the smooth embedding ι of M into $\mathbb{R}^k$ is also locally Lipschitz and, again by Lemma 2.0.69, the Hausdorff 2-measure of the image is zero:

$$\mathcal{H}^2_{\delta}(\mathrm{Im}(\iota \circ \mathbb{1}_M \circ H)) = 0.$$

On the other hand, note that the map $\iota \circ \mathbb{1}_M \circ H$ is a Lipschitz homotopy between the embeddings $\iota \circ \gamma_1$ and $\iota \circ \gamma_2$. Indeed, these are both smooth embeddings of $\mathbb{S}^1$ into $\mathbb{R}^k$, as ι is smooth, whose images differ since ι is an embedding and the images $\mathrm{Im}(\gamma_1) \neq \mathrm{Im}(\gamma_2)$ differ. By Lemma 7.0.6, the Hausdorff 2-measure of the image of the composition is positive,

$$\mathcal{H}^2_{\delta}(\mathrm{Im}(\iota \circ \mathbb{1}_M \circ H)) > 0,$$

a contradiction. $\square$

<u>Uncountably many homotopy classes in $\pi_1^{\mathrm{Lip}}(M, d_{CC}^M)$</u>

Similar to the π_1^{H} case, this is enough to show that, for any contact 3-manifold, $\pi_1^{\mathrm{Lip}}(M, d_{CC}^M)$ is uncountably generated. Corollary 7.0.7 will be used to construct an injective map of sets from the set of Legendrian knots into $\pi_1^{\mathrm{Lip}}(M, d_{CC}^M)$. Lemma 4.0.22 and Lemma **??** will then be used to gain the result.

Corollary 7.0.8. *There is an injective map of sets*

$$\left\{ \begin{array}{c} \text{Based, Oriented,} \\ \text{Legendrian Knots in } (M,\xi) \end{array} \right\} \hookrightarrow \pi_1^{Lip}(M,\xi)$$

$$K \longmapsto [\gamma]$$

where $\gamma : \mathbb{S}^1 \hookrightarrow (M,\xi)$ is a based, orientation-preserving horizontal embedding parametrizing the knot K.

Proof. First, the indicated map will be shown to be well-defined. Let K be a based, oriented, Legendrian knot in (M,ξ). By definition, there is at least one horizontal embedding $\gamma : \mathbb{S}^1 \hookrightarrow (M,\xi)$ parametrizing K. By Theorem 3.1 in [6], γ, and any other horizontal embedding of $\mathbb{S}^1$ into (M,ξ), is a Lipschitz map with respect to the Carnot-Carathéodory metric. Thus, $[\gamma]$ is indeed an element in the group $\pi_1^{\mathrm{Lip}}(M, d_{CC}^M)$.

Now, it is shown that different parametrizations of the same knot determine the same element in $\pi_1^{\mathrm{Lip}}(M, d_{CC}^M)$. Suppose that $\gamma, \psi : \mathbb{S}^1 \hookrightarrow (M,\xi)$ are horizontal embeddings such that $\mathrm{Im}(\gamma) = K = \mathrm{Im}(\psi)$ and each parametrization agrees with the orientation on K. To verify that this map is well-defined, it will be shown that γ and ψ are Lipschitz homotopic.

As ψ maps diffeomorphically onto the submanifold K,

$$\psi^{-1} : K \longrightarrow \mathbb{S}^1$$

is a smooth map that preserves orientation, as is the map $\psi^{-1} \circ \gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$. Thus, each arrow in the following commutative diagram is an orientation-preserving diffeomorphism,

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[dr, "\gamma", "\cong"'] \arrow[dd, dashed, "\psi^{-1}\circ\gamma"', "\cong"] & \\
& K. \\
\mathbb{S}^1 \arrow[ur, "\cong", "\psi"'] &
\end{tikzcd}$$

As $\psi^{-1} \circ \gamma$ is an orientation-preserving automorphism of $\mathbb{S}^1$, it is smoothly homotopic to the identity map on $\mathbb{S}^1$. Indeed, the degree of any orientation-preserving diffeomorphism is +1. The Hopf degree theorem guarantees that the self-maps $\psi^{-1} \circ \gamma$ and $\mathbb{1}_{\mathbb{S}^1}$ of $\mathbb{S}^1$ are smoothly homotopic. Let

$$h : [1,2] \times \mathbb{S}^1 \longrightarrow \mathbb{S}^1$$

be a smooth homotopy witnessing $\psi^{-1} \circ \gamma$ homotopic to $\mathbb{1}_{\mathbb{S}^1}$. As h is a smooth map between Riemannian manifolds whose domain is compact, h is Lipschitz. Then,

$$\psi \circ h : [1,2] \times \mathbb{S}^1 \longrightarrow (M, d_{CC}^M)$$

is a Lipschitz homotopy between the Lipschitz maps $\gamma = \psi \circ (\psi^{-1} \circ \gamma)$ and $\psi = \psi \circ \mathbb{1}_{\mathbb{S}^1}$.

So, the orientation-preserving maps γ and ψ paramterizing the knot K are Lipschitz homotopic. Since γ and ψ were arbitrary orientation-preserving parametrizations of K, the same Lipschitz homotopy class of $\pi_1^{\mathrm{Lip}}(M, d_{CC}^M)$ is

determined no matter the choice of orientation-preserving parametrization of a knot K. Thus, the map in the statement of the Corollary is well-defined.

To see that the map in the statement of the Corollary is injective, take distinct, based, oriented, Legendrian knots $K \neq K'$ in (M, ξ). Take parametrizations $\gamma, \gamma' : \mathbb{S}^1 \hookrightarrow (M, \xi)$ associated to the knots K and K', respectively. Thus, the images of the chosen parametrizations $\mathrm{Im}(\gamma) = K \neq K' = \mathrm{Im}(\gamma')$ are distinct and, by Corollary 7.0.7, each parametrization determines distinct Lipschitz homotopy classes, i.e., $[\gamma] \neq [\gamma']$. Thus, the map in the statement of the Corollary is injective.

□

Using the injective map defined in Corollary 7.0.8 and a few of the counting results from the π_1^{H} case, it will be shown that the group $\pi_1^{\mathrm{Lip}}(M, d_{CC}^M)$ is uncountably generated as well.

Theorem 7.0.9. *For any contact 3-manifold (M, ξ), the group $\pi_1^{Lip}(M, d_{CC}^M)$ is uncountably generated.*

Proof. Lemma 4.0.22 guarantees that there are uncountably many based knots in the contact 3-manifold (M, ξ). Thus, the domain of the injective map in Corollary 7.0.8 is uncountable and the map's target, $\pi_1^{\mathrm{Lip}}(M, d_{CC}^M)$ is therefore uncountable. By Lemma **??**, the group $\pi_1^{\mathrm{Lip}}(M, d_{CC}^M)$ is uncountably generated. □

A distributional open embedding induces an injective map on π_1^{Lip}

In this section, we will argue that a distributional open embedding of a contact 3-manifold into another contact 3-manifold induces an injective map between the respective first Lipschitz homotopy groups. This will follow from (M, d_{CC}^M) being purely 2-unrectifiable and Theorem 5 in [34]. Recall that this theorem states that

any Lipschitz map from a quasi-convex, Lipschitz simply-connected metric space into a purely 2-unrectifiable space factors through a metric tree.

Lemma 7.0.10. *Let (M, ξ, g) be a contact 3-manifold with an associated Carnot-Carathéodory metric d_{CC}^M. Let $H_0 : \mathbb{D}^2 \to (M, d_{CC}^M)$ be a Lipschitz map. Then, the map H_0 is homotopic to a Lipschitz map $H_1 : \mathbb{D}^2 \to (M, d_{CC}^M)$ such that the image of H_1 is contained in the image of the boundary of $\mathbb{D}^2$ under the initial map:* $\mathrm{Im}(H_1) = \mathrm{Im}(H_0|_{\partial \mathbb{D}^2})$. *Furthermore, the homotopy is relative to the boundary of $\mathbb{D}^2$.*

Proof. Let $H_0 : \mathbb{D}^2 \longrightarrow (M, d_{CC}^M)$ be a Lipschitz map. Recall that (M, d_{CC}^M) is purely 2-unrectifiable (Theorem 6.0.12). By Theorem 5 in [34], since $\mathbb{D}^2$ is quasi-convex and Lipschitz simply-connected, the map H_0 factors through a metric tree T:

$$\begin{tikzcd}
\mathbb{D}^2 \arrow[rr, "H_0"] \arrow[dr, "\psi"'] & & (M, d_{CC}^M). \\
& T \arrow[ur, "\phi"'] &
\end{tikzcd}$$

Both maps ψ and ϕ are Lipschitz. Since ψ is then continuous, the image $\psi(\partial \mathbb{D}^2) \subset T$ is connected and compact. So, $\psi(\partial \mathbb{D}^2)$ is a subtree of the metric tree T. By Lemma 6.0.19, there exists a Lipschitz deformation retract $F : I \times T \to T$ onto the subtree $\psi(\partial \mathbb{D}^2)$.

Consider the following diagram:

$$\begin{tikzcd}
\{1\} \times \mathbb{D}^2 \arrow[r, dashed] \arrow[d, hook] & \{1\} \times T \arrow[r, dashed] \arrow[d, hook] & \psi(\partial \mathbb{D}^2) \arrow[r, dashed] \arrow[d, hook] & \phi \circ \psi(\partial \mathbb{D}^2) \arrow[r, equal] \arrow[d, hook] & \mathrm{Im}(H_0|_{\partial \mathbb{D}^2}) \arrow[dl, hook'] \\
I \times \mathbb{D}^2 \arrow[r, "\mathbb{1} \times \psi"] & I \times T \arrow[r, "F"] & T \arrow[r, "\phi"] & M. & \\
\{0\} \times \mathbb{D}^2 \arrow[u, hook] \arrow[urrr, bend right, "H_0"] & & & &
\end{tikzcd}$$

The desired homotopy is $\phi \circ F \circ (\mathbb{1}, \psi)$. As is indicated in the diagram above, precomposing by the natural inclusion of $\{0\} \times \mathbb{D}^2$ into $I \times \mathbb{D}^2$ yields the original

map H_0. Precomposing by the natural inclusion of $\{1\} \times \mathbb{D}^2$ into $I \times \mathbb{D}^2$ yields a map that has image contained in $\mathrm{Im}(H_0|_{\partial\mathbb{D}^2}) \subset M$. Moreover, since the map F is a deformation retract onto $\psi(\partial\mathbb{D}^2)$, the homotopy $\phi \circ F \circ (\mathbb{1}, \psi)$ is constant on $\partial\mathbb{D}^2$ for all time $t \in I$.

□

Theorem 7.0.11. *Let (M, ξ, g) and (M', ξ', g') be contact 3-manifolds. Let $\varphi : (M, \xi) \hookrightarrow (M', \xi')$ be an open distributional embedding. Then the homomorphism induced by φ on first Lipschitz homotopy groups*

$$\varphi_\# : \pi_1^{Lip}(M, d_{CC}^M) \longrightarrow \pi_1^{Lip}(M', d_{CC}^{M'})$$

is injective.

Proof. As $\varphi_\#$ is a homomorphism, we can show that the map is injective by showing that the kernel of the map is trivial.

Let $\alpha : \mathbb{S}^1 \to (M, d_{CC}^M)$ represent an element of the kernel of $\varphi_\#$, that is, there exists a Lipschitz map $H : \mathbb{D}^2 \to (M', d_{CC}^{M'})$ such that H restricted to the boundary is the Lipschitz map $\varphi \circ \alpha$:

$$H|_{\partial\mathbb{D}^2} = \varphi \circ \alpha.$$

Note that the map $\varphi \circ \alpha$ is Lipschitz by Lemma 2.0.59.

By Lemma 7.0.10, the Lipschitz map H can be taken such that the image of H equals the image of the Lipschitz map $\varphi \circ \alpha$. Thus, H takes image entirely in the image of φ:

$$\mathrm{Im}(H) = \mathrm{Im}(\varphi \circ \alpha) \subset \mathrm{Im}(\varphi).$$

Since the inverse $\varphi^{-1} : \mathrm{Im}\,\varphi \to (M, \xi)$ is a distributional diffeomorphism, the map

given by composition

$$\varphi^{-1} \circ H : \mathbb{D}^2 \longrightarrow (M, d_{CC}^M)$$

is Lipschitz (Lemma 2.0.59) and, when the map is restricted to the boundary of $\mathbb{D}^2$ equals the map α. Thus, α is Lipschitz null homotopic. Thus, the only element in the kernel of $\varphi_\#$ is the trivial homotopy class.

□

Remark 7.0.12. Theorem 7.0.11 indicates that the cardinality of $\pi_1^{\mathrm{Lip}}(M, d_{CC}^M)$ is extremely large for any contact 3-manifold (M, ξ). For a base point in a contact 3-manifold, any connected, open neighborhood is a contact 3-manifold that openly embeds into (M, ξ). Thus, a copy of the uncountable set $\pi_1^{\mathrm{Lip}}(U, \xi|_U)$ is a subgroup of $\pi_1^{\mathrm{Lip}}(M, d_{CC}^M)$. Further, we suspect that each distinct neighborhood of a base point yields a distinct (i.e., not equal) uncountable subgroup of $\pi_1^{\mathrm{Lip}}(M, d_{CC}^M)$.

UNIVERSAL PATH SPACE OF CONTACT 3-MANIFOLDS

The goal of this chapter is to define and inspect a notion of a metric universal path space for $\mathbb{H}^1$ and in fact any contact 3-manifold with a sub-Riemannian structure. The universal path space construction will echo the construction of the universal cover for path-connected, locally path-connected, and semilocally simply connected spaces. We prove that this universal path space, much like the universal cover, will have a unique lifting property and be simply connected.

Let (M, ξ, g) be a contact 3-manifold endowed with a sub-Riemannian structure. As in previous chapters, this induces a Carnot-Carathéodory metric on M, denoted d^M_{CC}. Then, we will construct a metric space $\mathcal{P}_{(M,d^M_{CC})}$ called the *universal path space* of the metric space (M, d^M_{CC}) and a 1-Lipschitz map

$$\pi : \mathcal{P}_{(M,d^M_{CC})} \longrightarrow (M, d^M_{CC})$$

such that the fibers of this map are copies of the first Lipschitz homotopy group $\pi_1^{\mathrm{Lip}}(M, d^M_{CC})$.

Claim 8.0.0.1. Let N be a based, connected Riemannian manifold and let $f : N \to (M, d^M_{CC})$ be a locally Lipschitz, based map such that the induced map between homotopy groups

$$f_\# : \pi_1^{\mathrm{Lip}}(N) \longrightarrow \pi_1^{\mathrm{Lip}}(M, d^M_{CC})$$

is the constant homomorphism to the identity element of the group $\pi_1^{\mathrm{Lip}}(M, d^M_{CC})$. Then, there exists a unique, based, locally Lipschitz map

$$\begin{tikzcd} & \mathcal{P}_{(M,d_{CC}^M)} \arrow[d, "\pi"] \\ N \arrow[ur, dashed, "\exists!"] \arrow[r, "f"'] & (M, d_{CC}^M). \end{tikzcd}$$

When this is the case, $(\mathcal{P}_{(M,d_{CC}^M)}, d_{\mathcal{P}})$ is said to have the *unique lifting property.*

Claim 8.0.0.1 is verified within this chapter.

Claim 8.0.0.2. For contact 3-manifold (M, d_{CC}^M), the universal path space $\mathcal{P}_{(M,d_{CC}^M)}$ is a metric tree, and is thus Lipschitz contractible.

A corollary to these two claims would be that the Lipschitz homotopy group $\pi_n^{\mathrm{Lip}}(M, d_{CC}^M)$ is trivial for all $n > 1$. As this has already been shown in Chapter 6, the desire is that these claims apply to a more general class of metric spaces, namely sub-Riemannian manifolds with dimension 2 distribution. Thus, the higher Lipschitz homotopy groups of all such spaces would be trivial.

Non-singular points in a contact 3-manifold

We will show that all points in a contact 3-manifold are non-singular, i.e., the only equivalence class of loops in $\pi_1^{\mathrm{Lip}}((M, d_{CC}^M), p)$ that has a representative in every neighborhood of p is the trivial class. Via results in [4], we will be able to define a metric on the universal path space. Also, the topology on the fibers of the projection map π will have a Cantor set topology.

Regular points and non-singular points

The following definitions of regular points and non-singular points are metric versions of definitions that appear in [4].

Definition 8.0.1. For a metric space (X,d), a point $x \in X$ is a *regular point* if there exists an open neighborhood $U \subset X$ of the point x such that the map induced by inclusion,

$$\pi_1^{\mathrm{Lip}}(U,x) \longrightarrow \pi_1^{\mathrm{Lip}}(X,x),$$

is the trivial map. If every point of X is a regular point, the space X is said to be *semi-locally simply connected.*

Lemma 8.0.2. *For a contact 3-manifold (M,ξ), no point in M is a regular point.*

Proof. Let $p \in M$ and take an open neighborhood $U \subset M$ of the point p. By the biLipschitz Darboux theorem (Corollary 6.0.11), there exists a biLipschitz Darboux neighborhood

$$\varphi : (V, d_{CC}^{\mathbb{H}^1}) \hookrightarrow (M, d_{CC}^{M})$$

for some neighborhood $V \subset \mathbb{H}^1$ of the origin such that $\varphi(0) = p$ and the biLipschitz Darboux neighborhood $\varphi(V) \subset U$ is contained in U.

Since $(V, \xi^{std}|_V)$ is a contact 3-manifold, there exists a horizontal embedding $\gamma : \mathbb{S}^1 \hookrightarrow V$ through the origin (see Lemma 4.0.22). As the map φ is a distributional embedding, the composition $\varphi \circ \gamma$ is a Legendrian knot based at the point p. The horizontal embedding $\varphi \circ \gamma$ is then Lipschitz with respect to the Carnot-Carathéodory metrics (Theorem 3.1 in [6]).

So, the Lipschitz loop $\varphi \circ \gamma$ represents an element of the first Lipschitz homotopy group $\pi_1^{\mathrm{Lip}}(U,p)$. By Corollary 7.0.7, this loop is not Lipschitz null-homotopic in U. But, also by Corollary 7.0.7, the loop $incl_U \circ \varphi \circ \gamma$ is not Lipschitz null-homotopic in M either. Thus, the map of first Lipschitz homotopy groups induced by inclusion of the open set U is not the trivial map.

$\square$

Definition 8.0.3. For a metric space (X, d), a point $x \in X$ is a *non-singular point* if only the trivial homotopy class can be represented by a loop within each neighborhood of x, that is, for the class $[\gamma] \in \pi_1^{\mathrm{Lip}}(X, x)$, if for any open neighborhood $U \subset X$ of the point x, there exists a Lipschitz loop $\gamma_U : \mathbb{S}^1 \to U$ such that $\gamma_U \in [\gamma]$, then $[\gamma]$ is the trivial homotopy class, $[\gamma] = [x]$.

We will show that every point in a contact 3-manifold is non-singular. By the biLipschitz Darboux theorem, we only need to show that the origin in $\mathbb{H}^1$ is non-singular. The remainder of this section is dedicated to showing that the origin in $\mathbb{H}^1$ is non-singular. This will be accomplished by viewing $\mathbb{H}^1$ as an inverse limit of spaces where a compact neighborhood of the origin is collapsed.

Inverse systems of quotients of sub-Riemannian manifolds

Let (M, ξ, g) be a sub-Riemannian manifold. Denote as always the Carnot-Carathéodory metric on M by d_{CC}^M. Fix a collection of connected submanifolds with connected and compact boundary $N_n \subset M$ such that $M \setminus N_n$ is connected and the submanifolds are nested: $N_{n+1} \subset N_n$ for each $n \in \mathbb{N}$. Furthermore, assume that the intersection of the N_n is a submanifold. Denote the intersection by

$$N_\infty := \bigcap_{n=1}^{\infty} N_n.$$

The reader is encouraged to consider the manifold M as $\mathbb{H}^1$ and the submanifolds N_n as the closed disk in $\mathbb{H}^1$ centered at the origin that has radius $1/n$. Later, the subset N_∞ can be thought of as the origin. This construction will allow for homotopies that map into any neighborhood of the origin being Lipschitz extended to the origin.

We will define a sequence of metric spaces by collapsing the submanifold N_n in M to a point. For each $n \in \mathbb{N} \cup \{\infty\}$, let the associated equivalence relation be denoted

by $\sim_n$. Here, two points $p, p' \in M$ are equivalent $p \sim_n p'$ if the points are equal $p = p'$ or they are both in the submanifold $p, p' \in N_n$.

This results in a metric space M/N_n where the metric is given by the following: for $[p], [q] \in M/N_n$,

$$d^n([p],[q]) := \inf\left\{d^M_{CC}(p_1, q_1) + \ldots + d^M_{CC}(p_k, q_k)\right\},$$

where the infimum is taken over all finite sequences $(p_1, \ldots, p_k), (q_1, \ldots, q_k)$ of points in M such that

$$p \sim p_1 \ , \ q_1 \sim p_2 \ , \ \ldots \ , \ q_{k-1} \sim p_n \ , \ q_k \sim q.$$

Due to the rather simple nature of the equivalence relation on M and that N_n has compact boundary, the metric can be written in a simpler fashion.

Observation 8.0.4. Let $n \in \mathbb{N} \cup \{\infty\}$. For $[p], [q] \in M/N_n$,

$$d^n([p],[q]) = \min\left\{d^M_{CC}(p,q), d^M_{CC}(p, N_n) + d^M_{CC}(q, N_n)\right\}.$$

This follows from the submanifold ∂N_n being compact, $M \smallsetminus N_n$ being connected, and the real-valued function $d^M_{CC}(-, N_n)$ returning the distance to the submanifold being continuous. Thus, for any point p not in N_n, there is a closest point to p in ∂N_n.

We will show that the various quotient maps are Lipschitz. In fact, these maps are metric maps. We will say that a map is a *metric map* if its Lipschitz constant is bounded by 1. Before proceeding, we will make note of the universal property of metric quotients.

Observation 8.0.5. Let (X, d^X) be a metric space. For a metric map $f : (M, d^M_{CC}) \to (X, d^X)$ such that f is constant of equivalence classes with respect to $\sim_n$, there is a unique filler

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "Q_n"'] & X \\ M/N_n \arrow[ur, dashed, "\tilde{f}", "\exists!"'] & \end{tikzcd}$$

such that the map $\tilde{f}$ is a metric map. A continuous map $\tilde{f}$ is guaranteed to exist via the universal property of quotient spaces. That the map is in fact metric follows from f being metric and the definition of the quotient metric on M/N_n. In fact, this was not special to the case at hand. Rather, this is the universal property for any metric quotient space.

Lemma 8.0.6. *For $n \in \mathbb{N} \cup \{\infty\}$, the quotient map*

$$Q_n : M \longrightarrow M/N_n$$

is a metric map.

Proof. This is more-or-less clear from Observation 8.0.4. For points p and q in M,

$$\begin{aligned} d^n(Q_n(p), Q_n(q)) &= d^n([p],[q]) \\ &= \min\left\{d^M_{CC}(p,q), d^M_{CC}(p,N_n) + d^M_{CC}(q,N_n)\right\} \\ &\leq d^M_{CC}(p,q). \end{aligned}$$

∎

Lemma 8.0.7. *For any $n \in \mathbb{N}$, there is a quotient map*

$$q_n : M/N_{n+1} \longrightarrow M/N_n$$

that is a metric map and such that there is an equality of maps $Q_n \circ Q_{n+1} = q_n$.

Proof. Consider the quotient map $Q_n : M \to M/N_n$. We will argue that the map Q_n is constant on equivalence classes with respect to $\sim_{n+1}$ on M.

Let p and q be points in M such that they are equivalent with respect to $\sim_{n+1}$. So, either $p = q$ or both points reside in the submanifold N_{n+1}. If the points are equal, obviously Q_n sends them both to the same point in M/N_n. If both points are in the N_{n+1}, then they also are in N_n, since the submanifolds are nested. Thus, the points p and q are both sent to the point $N_n \in M/N_n$ by the map Q_n.

Thus, by the universal property of metric quotient spaces, there exists a unique metric map q_n such that the following diagram commutes,

$$\begin{tikzcd}
M \arrow[r, "Q_n"] \arrow[d, "Q_{n+1}"'] & M/N_n. \\
M/N_{n+1} \arrow[ur, dashed, "\exists!", "q_n"'] &
\end{tikzcd}$$

□

So, we have a diagram of metric spaces where each arrow is a metric map,

$$\ldots \xrightarrow{q_{n+1}} M/N_{n+1} \xrightarrow{q_n} M/N_n \xrightarrow{q_{n-1}} \ldots \xrightarrow{q_1} M/N_1.$$

We will now add a few additional assumptions on the collection of submanifolds $\{N_n\}$. Assume going forward that each N_n is compact and their intersection N_∞ is non-empty.

We will now show that, under these additional assumptions, the metric space M/N_∞ is the inverse limit of this inverse system

Lemma 8.0.8. *For each $n \in \mathbb{N}$, there exists a metric map $c_n : M/N_\infty \to M/N_n$ such that the following diagram commutes:*

$$\begin{tikzcd} M/N_\infty \arrow[d, "c_{n+1}"] \arrow[dr, "c_n"] & \\ M/N_{n+1} \arrow[r, "q_n"] & M/N_n. \end{tikzcd}$$

Proof. Consider the quotient map $Q_n : M \to M/N_n$. We will argue that the map Q_n is constant on equivalence classes with respect to $\sim_\infty$ on M.

Let p and q be points in M such that they are equivalent with respect to $\sim_\infty$. So, either $p = q$ or both points reside in the submanifold N_∞. If the points are equal, obviously Q_n sends them both to the same point in M/N_n. If both points are in the N_∞, then they also are in N_n, since the submanifolds are nested. Thus, the points p and q are both sent to the point $N_n \in M/N_n$ by the map Q_n.

Thus, by the universal property of metric quotient spaces, there exists a unique metric map c_n such that the following diagram commutes,

$$\begin{tikzcd} M \arrow[r, "Q_n"] \arrow[d, "Q_\infty"'] & M/N_n. \\ M/N_\infty \arrow[ur, dashed, "\exists!"', "c_n"] & \end{tikzcd}$$

Moreover, since the maps $q_n \circ Q_{n+1} = Q_n$ are equal by Lemma 8.0.7 and each map c_n is the unique filler, the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "Q_{n+1}"] \arrow[rr, bend left, "Q_n"] \arrow[d, "Q_\infty"'] & M/N_{n+1} \arrow[r, "q_n"] & M/N_n. \\ M/N_\infty \arrow[ur, dashed, "c_{n+1}"] \arrow[urr, dashed, "c_n"] & & \end{tikzcd}$$

$\square$

Thus, we have a diagram of metric spaces where each arrow is a metric map,

$$M/N_\infty \longrightarrow \dots \xrightarrow{q_{n+1}} M/N_{n+1} \xrightarrow{q_n} M/N_n \xrightarrow{q_{n-1}} \dots \xrightarrow{q_1} M/N_1. \qquad (8.1)$$

Note that Observation 8.0.4 yields a relationship between the metrics d^n for $n \in \mathbb{N}$ and d^∞.

Lemma 8.0.9. *For points $p, q \in M/N_\infty$,*

$$d^\infty(p,q) = \sup_{n\geq 1} d^n(c_n(p), c_n(q)).$$

Proof. Consider the equalities from Observation 8.0.4,

$$d^n(c_n(p), c_n(q)) = \min\left\{d^M_{CC}(p,q), d^M_{CC}(p,N_n) + d^M_{CC}(q,N_n)\right\},$$

and

$$d^\infty(p,q) = \min\left\{d^M_{CC}(p,q), d^M_{CC}(p,N_\infty) + d^M_{CC}(q,N_\infty)\right\}.$$

If p and q represent elements of the submanifold N_∞, then $d^n(c_n(p), c_n(q)) = 0$ for all $n \in \mathbb{N}$ and $d^\infty(p,q) = 0$.

Suppose that p is not in N_∞ while q is in N_∞. The sequence of real values $(d^M_{CC}(p,N_n))_{n\in\mathbb{N}}$ is increasing and is bounded above by $d^M_{CC}(p,N_\infty)$ due to the submanifolds N_n being nested. Thus, the sequence converges. In fact, this sequence converges to $d^M_{CC}(p,N_\infty)$ since each value $d^M_{CC}(p,N_n)$ has an associated point $p_n \in N_n$ that realizes the distance. The sequence (p_n) has a (sub)sequence that converges since N_1 is compact and the point of convergence must be in each N_n since each N_n is compact. Therefore, the sequence (p_n) converges to a point in N_∞. Thus, we have

an equality of values,

$$\sup_{n\geq 1} d^M_{CC}(p, N_n) = \lim_{n\to\infty} d^M_{CC}(p, N_n) = d^M_{CC}(p, N_\infty).$$

The desired equality then follows:

$$\begin{aligned} \sup_{n\geq 1} d^n(c_n(p), c_n(q)) &= \sup_{n\geq 1} \min\left\{d^M_{CC}(p,q), d^M_{CC}(p, N_n)\right\} \\ &= \min\left\{d^M_{CC}(p,q), d^M_{CC}(p, N_\infty)\right\} \\ &= d^\infty(p,q). \end{aligned}$$

The equality holds via a similar argument for when p and q both represent elements on the complement of N_∞.

□

We now show that M/N_∞ is the inverse limit of the inverse sequence (M/N_n). To accomplish this, let (A, d^A) be a metric space and $f_n : A \to M/N_n$ be L_n-Lipschitz maps such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[d, dashed, "f"] \arrow[rr, bend left, "f_{n+1}"] \arrow[rrr, bend left, "f_n"] \arrow[rrrrr, bend left, "f_1"] & & & & & \\ M/N_\infty \arrow[r] & \dots \arrow[r, "q_{n+1}"] & M/N_{n+1} \arrow[r, "q_n"] & M/N_n \arrow[r] & \dots \arrow[r, "q_1"] & M/N_1. \end{tikzcd} \tag{8.2}$$

We will show that such a Lipschitz map f exists and that it is the unique such map.

Since diagram (8.2) commutes, we can write down a formula for a map f if it were to exist in terms of the maps f_n. Let $a \in A$ and consider its image $f(a)$ in

M/N_∞. Suppose that $f(a)$ is not in the collapsed submanifold N_∞. Since the above diagram commutes and the map q_n fixes elements outside of the closed set N_n/N_{n+1}, there exists a positive integer K such that, for all $n \geq K$, the map f_n evaluated at a determines the same point:

$$q_K \circ \ldots \circ q_{n-1} \circ f_n(a) = f_K(a).$$

For each $n \geq K$, the point $f_n(a)$ is in $(M/N_n) \smallsetminus (N_n/N_n)$ and thus is associated to a point in M. So, for a point $a \in A$ that does not get sent to N_∞/N_∞, the image is determined by the Kth map: $f(a) = f_K(a)$.

Note that, if a does get sent to the collapsed submanifold N_∞ by f, then $f_n(a)$ is in N_∞ for all n. Thus, we have described a map f which would fill diagram (8.2):

$$f(a) = \begin{cases} f_K(a) & \text{, if } \exists K \text{ s.t. } f_K(a) \notin N_\infty \\ N_\infty/N_\infty & \text{, else.} \end{cases}$$

In fact, we have written down the only possible map that could fill this diagram. We now go about showing that f is Lipschitz.

Lemma 8.0.10. *For each n, the Lipschitz constant L_n for the Lipschitz map f_n is less than or equal to the Lipschitz constant L_{n+1} associated to the Lipschitz map f_{n+1}.*

Proof. Since diagram (8.2) commutes, we have an equality of Lipschitz maps $q_n \circ f_{n+1} = f_n$. So, for distinct points x and y in the metric space A, we have the following inequality:

$$\begin{aligned} d^n(f_n(x), f_n(y)) &= d^n(q_n \circ f_{n+1}(x), q_n \circ f_{n+1}(y)) \\ &\leq d^{n+1}(f_{n+1}(x), f_{n+1}(y)). \end{aligned}$$

Thus, the following inequality holds:

$$L_n = \sup_{x \neq y} \frac{d^n(f_n(x), f_n(y))}{d^A(x,y)} \leq \sup_{x \neq y} \frac{d^{n+1}(f_{n+1}(x), f_{n+1}(y))}{d^A(x,y)} = L_{n+1}.$$

$\square$

Lemma 8.0.11. *For any point $p \in M$ and any $n \in \mathbb{N}$,*

$$d_{CC}^M(p, N_n) \leq d_{CC}^M(p, N_1) + |N_1|.$$

Proof. As N_n is compact, there exists a point $p' \in N_n$ that witnesses the distance from p to N_n,

$$d_{CC}^M(p, N_n) = d_{CC}^M(p, p').$$

Also, as N_1 is compact, there exists a point $p'' \in N_1$ that witnesses the distance from p to N_1,

$$d_{CC}^M(p, N_1) = d_{CC}^M(p, p'').$$

By the triangle inequality,

$$d_{CC}^M(p, p') \leq d_{CC}^M(p, p'') + d_{CC}^M(p'', p').$$

Since p' and p'' are in N_1, the distance between them is bounded by the diameter of N_1. Thus, we have the desired inequality:

$$d_{CC}^M(p, N_n) = d_{CC}^M(p, p') \leq d_{CC}^M(p, p'') + d_{CC}^M(p'', p') \leq d_{CC}^M(p, N_1) + |N_1|.$$

$\square$

Lemma 8.0.12. *For* $[p],[q] \in N_n$*,*

$$d^n([p],[q]) \le d^1(q_1 \circ \dots q_{n-1}[p], q_1 \circ \dots q_{n-1}[q]) + 2|N_1|.$$

Proof. This follows immediately from Lemma 8.0.11:

$$\begin{aligned} d^n([p],[q]) &= \min\left(d^M_{CC}(p,q), d^M_{CC}(p,N_n) + d^M_{CC}(q,N_n)\right) \\ &\le \min\left(d^M_{CC}(p,q), d^M_{CC}(p,N_1) + d^M_{CC}(q,N_1) + 2|N_1|\right) \\ &\le \min\left(d^M_{CC}(p,q), d^M_{CC}(p,N_1) + d^M_{CC}(q,N_1)\right) + 2|N_1| \\ &= d^1(q_1 \circ \dots q_{n-1}[p], q_1 \circ \dots q_{n-1}[q]) + 2|N_1|. \end{aligned}$$

□

Lemma 8.0.13. *The sequence of Lipschitz constant* (L_n) *is bounded above. Thus, the sequence of Lipschitz constants converges.*

Proof. This follows from the previous observations and lemmas as well as the definition of the Lipschitz constant:

$$\begin{aligned} L_n &:= \sup_{x \ne y} \frac{d^n(f_n(x), f_n(y))}{d^A(x,y)} && \text{(Definition)} \\ &\le 2|N_1| \\ &\quad + \sup_{x \ne y} \frac{d^1(q_1 \circ \dots \circ q_{n-1} \circ f_n(x), q_1 \circ \dots \circ q_{n-1} \circ f_n(y))}{d^A(x,y)} && \text{(Lemma 8.0.12)} \\ &= 2|N_1| + \sup_{x \ne y} \frac{d^1(f_1(x), f_1(y))}{d^A(x,y)} \\ &= 2|N_1| + L_1. \end{aligned}$$

So, the sequence (L_n) is bounded above by $2|N_1| + L_1$. By Lemma 8.0.10, the sequence is also monotonically increasing. Thus, the sequence converges. □

Lemma 8.0.14. *The map* $f : A \longrightarrow M/N_\infty$ *is Lipschitz*

Proof. We use the definition of the Lipschitz constant with respect to the map f.

$$\begin{aligned}
\sup_{x\neq y}\frac{d^{\infty}(f(x),f(y))}{d^{A}(x,y)} &= \sup_{x\neq y}\frac{\sup_{n\geq 1} d^{n}(f_n(x),f_n(y))}{d^{A}(x,y)} && \text{(Lemma 8.0.9)}\\
&= \sup_{x\neq y}\sup_{n\geq 1}\frac{d^{n}(f_n(x),f_n(y))}{d^{A}(x,y)}\\
&= \sup_{n\geq 1}\sup_{x\neq y}\frac{d^{n}(f_n(x),f_n(y))}{d^{A}(x,y)}\\
&= \sup_{n\geq 1} L_n && \text{(Lemma 8.0.13)}\\
&< \infty.
\end{aligned}$$

Thus, the map f has a Lipschitz constant and is thus Lipschitz.

□

Thus, the Lipschitz map f fills the inverse system (8.2). It is the unique such map. This confirms the following.

Corollary 8.0.15. *Let (M, d_{CC}^{M}) be a sub-Riemannian manifold and let $\{N_n\}$ be a collection of nested, connected, compact submanifolds with boundary such that the intersection $N_\infty := \cap_{n=1}^{\infty} N_n$ is a non-empty submanifold. Then, the quotient space M/N_∞ is the metric inverse limit of the inverse system*

$$M/N_\infty \longrightarrow \dots \xrightarrow{q_{n+1}} M/N_{n+1} \xrightarrow{q_n} M/N_n \xrightarrow{q_{n-1}} \dots \xrightarrow{q_1} M/N_1.$$

The following corollaries follow immediately from Corollary 8.0.15.

Corollary 8.0.16. *Let M be a sub-Riemannian manifold that is complete with respect to d_{CC}^{M}. Let $p \in M$ be any point in M and denote the closed ball of radius $\frac{1}{n}$ centered at p by $\mathbb{D}_n$. Then,*

$$M \cong \lim_{n\geq 1} M/\mathbb{D}_n.$$

In particular, when M is taken to be the first Heisenberg group and p is the origin,

$$\mathbb{H}^1 \cong \lim_{n\geq 1} \mathbb{H}^1/\mathbb{D}_n.$$

Proof. Apply Corollary 8.0.15 to M with the nested collection of compact balls $\{\mathbb{D}_n\}$. The intersection of these compacts balls is the singleton $\{p\}$. Thus, $M/\{p\} \cong \lim_{n\geq 1} M/\mathbb{D}_n$.

There is an obvious bijection from $M/\{p\}$ to M as the indicated quotient is collapsing a single point. Further, it is clear from Observation 8.0.4 that the bijection is an isometry. □

Corollary 8.0.17. *Endow the manifold* $\mathbb{S}^1 \times [0,1]$ *with its standard Riemannian structure and let* d_e *denote the path metric. The following spaces are isomorphic,*

$$\mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times \{0\} \cong \lim_{n\geq 1} \mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times \left[0, \frac{1}{n}\right].$$

Proof. Apply Corollary 8.0.15 to $\mathbb{S}^1 \times [0,1]$ with the nested collection of compact sets $\{\mathbb{S}^1 \times \left[0, \frac{1}{n}\right]\}$. □

Observation 8.0.18. There is a Lipschitz map from the 2-disk to the left-hand side of the isomorphism in Corollary 8.0.17 given by polar coordinates:

$$\begin{tikzcd}
\mathbb{D}^2 \arrow[rr] & & \mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times \{0\}. \\
& \mathbb{S}^1 \arrow[ul, hook, "\partial"] \arrow[ur, hook, "(\mathbb{1},1)"'] &
\end{tikzcd}$$

This map is in fact as isomorphism since $\mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times \{0\}$ is the cone on $\mathbb{S}^1$.

We will now go about showing that the inverse system $\left(\mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times \left[0, \frac{1}{n}\right]\right)$ is equivalent to the inverse system $\left(\mathbb{S}^1 \times (0,1]/\mathbb{S}^1 \times \left(0, \frac{1}{n}\right]\right)$. Thus, the quotient $\mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times \{0\}$ is also the inverse limit to this to-be-defined inverse system.

For $n \in \mathbb{N}$, consider the quotient space $\mathbb{S}^1 \times (0,1]/\mathbb{S}^1 \times \left(0, \frac{1}{n}\right]$. Endow the space $\mathbb{S}^1 \times (0,1]$ with its standard Riemannian structure and denote the path metric by d_e. This choice of notation is reasonable as the path metric on $\mathbb{S}^1 \times (0,1]$ agrees with the restricted path metric on $\mathbb{S}^1 \times [0,1]$.

There is a metric on the quotient space defined as before, denoted d_e^n. Observation 8.0.4, Lemma 8.0.6, and Lemma 8.0.7 did not depend on the collection of submanifolds $\{N_n\}$ being compact. So, Observation 8.0.4 yields that the metric d_e^n can be described as follows: For $[p],[q] \in \mathbb{S}^1 \times (0,1]/\mathbb{S}^1 \times \left(0,\frac{1}{n}\right]$,

$$d_e^n([p],[q]) = \min\left\{d_e(p,q), d_e\left(p, \mathbb{S}^1 \times \left(0,\frac{1}{n}\right]\right) + d_e\left(q, \mathbb{S}^1 \times \left(0,\frac{1}{n}\right]\right)\right\}.$$

Lemma 8.0.6 and Lemma 8.0.7 yield that the natural maps between the quotient spaces $\mathbb{S}^1 \times \left(0,\frac{1}{n}\right]$ are metric maps. Thus, we have an inverse system of metric spaces with metric maps:

$$\ldots \longrightarrow \mathbb{S}^1 \times (0,1]/\mathbb{S}^1 \times \left(0,\tfrac{1}{n+1}\right] \longrightarrow \mathbb{S}^1 \times (0,1]/\mathbb{S}^1 \times \left(0,\tfrac{1}{n}\right] \longrightarrow \ldots. \tag{8.3}$$

We will show that this inverse system is isomorphic to the inverse system,

$$\ldots \longrightarrow \mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times \left[0,\tfrac{1}{n+1}\right] \longrightarrow \mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times \left[0,\tfrac{1}{n}\right] \longrightarrow \ldots. \tag{8.4}$$

Since the inverse systems (8.3) and (8.4) are isomorphic and the inverse limit of (8.4) is isomorphic to $\mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times \{0\}$, the inverse limit of (8.3) exists and is also isomorphic to this space.

We will denote the metric on $\mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times [0,\frac{1}{n}]$ by d_i^n so as to distinguish this metric from the metric on $\mathbb{S}^1 \times (0,1]/\mathbb{S}^1 \times (0,\frac{1}{n}]$, which is denoted d_e^n.

Lemma 8.0.19. *The inverse systems* (8.3) *and* (8.4) *are isomorphic. Thus, for the inverse system* (8.3)*, the metric inverse limit exists and is isomorphic to the inverse*

limit of (8.4)*:*

$$\lim_{n\geq 1} \mathbb{S}^1 \times (0,1]/\mathbb{S}^1 \times \left(0, \frac{1}{n}\right] \cong \lim_{n\geq 1} \mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times \left[0, \frac{1}{n}\right] \cong \mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times \{0\}.$$

Proof. First, we will see that there are natural maps from the inverse system (8.3) to the inverse system (8.4) induced by inclusion. Indeed, for $m > n$, there is a commutative diagram given by inclusions,

$$\begin{tikzcd}
\mathbb{S}^1 \times (0,1] \arrow[r, hook] & \mathbb{S}^1 \times [0,1] \\
\mathbb{S}^1 \times \left(0, \frac{1}{n}\right] \arrow[r, hook] \arrow[u, hook] & \mathbb{S}^1 \times \left[0, \frac{1}{n}\right] \arrow[u, hook] \\
\mathbb{S}^1 \times \left(0, \frac{1}{m}\right] \arrow[r, hook] \arrow[u, hook] & \mathbb{S}^1 \times \left[0, \frac{1}{m}\right] \arrow[u, hook],
\end{tikzcd}$$

which induces a commutative diagram of quotient spaces,

$$\begin{tikzcd}
\mathbb{S}^1 \times (0,1]/\mathbb{S}^1 \times \left(0, \frac{1}{m}\right] \arrow[r] \arrow[d] & \mathbb{S}^1 \times (0,1]/\mathbb{S}^1 \times \left(0, \frac{1}{n}\right] \arrow[d] \\
\mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times \left[0, \frac{1}{m}\right] \arrow[r] & \mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times \left[0, \frac{1}{n}\right].
\end{tikzcd}$$

As will be argued, each such map down,

$$\mathbb{S}^1 \times (0,1]/\mathbb{S}^1 \times \left(0, \frac{1}{n}\right] \longrightarrow \mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times \left[0, \frac{1}{n}\right],$$

is an isomorphism. Indeed, the collapse of each space by the quotient leaves a copy of $\mathbb{S}^1 \times \left(\frac{1}{n}, 1\right]$ away from the single collapsed point. Thus, this maps acts like the identity away from the collapsed point and sends the collapsed point to the collapsed point. In fact, the spaces are isometric since the quotient metrics on each arise from the

same metric on the space $\mathbb{S}^1 \times [0,1]$ before collapsing and, for any point $p \in \mathbb{S}^1 \times [0,1]$,

$$d_e\left(p, \mathbb{S}^1 \times \left(0, \frac{1}{n}\right]\right) = d_e\left(p, \mathbb{S}^1 \times \left[0, \frac{1}{n}\right]\right).$$

Thus, there is a map of inverse systems where each map between the systems is an isomorphism,

$$\begin{tikzcd}
\dots \arrow[r] & \mathbb{S}^1 \times (0,1]/\mathbb{S}^1 \times \left(0, \frac{1}{n+1}\right] \arrow[r] \arrow[d, "\cong"] & \mathbb{S}^1 \times (0,1]/\mathbb{S}^1 \times \left(0, \frac{1}{n}\right] \arrow[r] \arrow[d, "\cong"] & \dots \\
\dots \arrow[r] & \mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times \left[0, \frac{1}{n+1}\right] \arrow[r] & \mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times \left[0, \frac{1}{n}\right] \arrow[r] & \dots,
\end{tikzcd}$$

Since the inverse limit of the bottom inverse system exists (Corollary 8.0.17), the inverse limit of the top inverse system exists and is isomorphic to $\mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times \{0\}$. □

The origin in $\mathbb{H}^1$ is non-singular

We will now show that the origin in $\mathbb{H}^1$ is a non-singular point. We will assume that there is some homotopy class in $\pi_1^{\mathrm{Lip}}(\mathbb{H}^1, 0)$ that has a representative in every neighborhood of the origin. We will take a representative of the homotopy class and look at the nested neighborhoods $\mathbb{D}_n$ to find homotopic loops. From these loops, we will be able to glue associated homotopies together into a map out of $\mathbb{S}^1 \times (0,1]$ into $\mathbb{H}^1$. Via Lemma 8.0.19, we will be able to argue that this can be extended to a Lipschitz null-homotopy of the original representative.

This will be enough to show that any point in any contact 3-manifold is a non-singular point by the biLipschitz Darboux theorem.

Theorem 8.0.20. *The origin in the first Heisenberg group $\mathbb{H}^1$ is a non-singular point.*

Proof. Let $[\gamma_0] \in \pi_1^{\mathrm{Lip}}(\mathbb{H}^1, 0)$ be a homotopy class such that every neighborhood of the origin contains a representative of the class $[\gamma_0]$. Let $\gamma_0 : \mathbb{S}^1 \longrightarrow \mathbb{H}^1$ be a Lipschitz

loop in this homotopy class. By assumption, for each $n \in \mathbb{N}$, there is a representative $\gamma_n : \mathbb{S}^1 \longrightarrow \mathbb{D}_n$ that is homotopic to γ_{n-1} via a Lipschitz homotopy

$$H_n : \mathbb{S}^1 \times \left[\frac{1}{n+1}, \frac{1}{n}\right] \longrightarrow \mathbb{D}_{n-1}$$

where $H_n\left(-, \frac{1}{n+1}\right) = \gamma_n$ and $H_n\left(-, \frac{1}{n}\right) = \gamma_{n-1}$. Let $\mathbb{D}_0$ be the entire space $\mathbb{H}^1$. We can proceed inductively to glue these Lipschitz homotopies together to get a Lipschitz homotopy from γ_0 to γ_n:

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[r, hook, "(\mathbb{1},1)"] \arrow[d, "\gamma_0"] & \mathbb{S}^1 \times \left[\frac{1}{n+1}, 1\right] \arrow[dl, dashed] & & \mathbb{S}^1 \times \left[\frac{1}{n+1}, \frac{1}{n}\right] \arrow[ll, hook'] \arrow[d, "H_n"] & \mathbb{S}^1 \arrow[l, hook', "\left(\mathbb{1}, \frac{1}{n+1}\right)"'] \arrow[d, "\gamma_n"] \\
\mathbb{H}^1 & & & \mathbb{D}_{n-1} \arrow[lll, hook'] & \mathbb{D}_n. \arrow[l, hook']
\end{tikzcd}$$

Continuing to glue all such homotopies together, we construct a map

$$H : \mathbb{S}^1 \times (0, 1] \longrightarrow \mathbb{H}^1$$

witnessing all of the chosen homotopies:

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[r, hook, "(\mathbb{1},1)"] \arrow[d, "\gamma_0"] & \mathbb{S}^1 \times (0, 1] \arrow[dl, dashed, "H"', "\exists"] & \mathbb{S}^1 \times \left(0, \frac{1}{2}\right] \arrow[l, hook'] \arrow[d, "\exists"] & \mathbb{S}^1 \times \left(0, \frac{1}{3}\right] \arrow[l, hook'] \arrow[d, "\exists"] & \dots \arrow[l, hook'] \\
\mathbb{H}^1 & & \mathbb{D}_1 \arrow[ll, hook'] & \mathbb{D}_2 \arrow[l, hook'] & \dots . \arrow[l, hook']
\end{tikzcd}$$

For each sub-diagram,

$$\begin{tikzcd}
\mathbb{S}^1 \times (0, 1] \arrow[d, "H"] & \mathbb{S}^1 \times \left(0, \frac{1}{n+1}\right] \arrow[l, hook'] \arrow[d] \\
\mathbb{H}^1 & \mathbb{D}_n, \arrow[l, hook']
\end{tikzcd}$$

there is an induced map on the quotient space coming from collapsing the subspace $\mathbb{S}^1 \times \left(0, \frac{1}{n+1}\right]$ within $\mathbb{S}^1 \times (0, 1]$:

$$\begin{tikzcd}
\mathbb{S}^1 \times (0,1] \arrow[r] \arrow[d, "H"] \arrow[dr, "c_n \circ H"] & \mathbb{S}^1 \times (0,1]/\mathbb{S}^1 \times \left(0, \frac{1}{n+1}\right] \arrow[d, dashed, "\exists!"', "H^n"] \\
\mathbb{H}^1 \arrow[r, "c_n"] & \mathbb{H}^1/\mathbb{D}_n.
\end{tikzcd}$$

In fact, each map H^n is Lipschitz since the map H is Lipschitz on $\mathbb{S}^1 \times \left[\frac{1}{n+1}, 1\right]$. Moreover, for $m > n$, we have the commutative diagram

$$\begin{tikzcd}
\mathbb{S}^1 \times (0,1] \arrow[r] \arrow[d, "H"] & \mathbb{S}^1 \times (0,1]/\mathbb{S}^1 \times \left(0, \frac{1}{m+1}\right] \arrow[r] \arrow[d, "H^m"', "\exists!"] & \mathbb{S}^1 \times (0,1]/\mathbb{S}^1 \times \left(0, \frac{1}{n+1}\right] \arrow[d, "H^n"', "\exists!"] \\
\mathbb{H}^1 \arrow[r, "c_m"] \arrow[rr, bend right, "c_n"'] & \mathbb{H}^1/\mathbb{D}_m \arrow[r] & \mathbb{H}^1/\mathbb{D}_n.
\end{tikzcd} \tag{8.5}$$

Thus, the diagrams (8.5) compile into a diagram:

$$\begin{tikzcd}
\mathbb{S}^1 \times (0,1] \arrow[r] \arrow[d, "H"] & \dots \arrow[r] & \mathbb{S}^1 \times (0,1]/\mathbb{S}^1 \times \left(0, \frac{1}{n+1}\right] \arrow[r] \arrow[d, "H^n"] & \dots \\
\mathbb{H}^1 \arrow[r] & \dots \arrow[r] & \mathbb{H}^1/\mathbb{D}_n \arrow[r] & \dots.
\end{tikzcd} \tag{8.6}$$

Since $\mathbb{H}^1$ is the inverse limit of the bottom inverse system (Corollary 8.0.16), the map H is the unique map from $\mathbb{S}^1 \times (0,1]$ to $\mathbb{H}^1$ such that this diagram commutes. Thus, we get the following diagram:

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[d, hook, "(\mathbb{1},1)"] & & & & \\
\mathbb{S}^1 \times (0,1] \arrow[d, dashed, "b"', "\exists!"] \arrow[drr, bend left] & & & & \\
\mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times \{0\} \arrow[r] \arrow[d, dashed, "a"', "\exists!"] & \dots \arrow[r] & \mathbb{S}^1 \times (0,1]/\mathbb{S}^1 \times \left(0, \frac{1}{n+1}\right] \arrow[r] \arrow[d, "H^n"] & \dots \\
\mathbb{H}^1 \arrow[r] & \dots \arrow[r] & \mathbb{H}^1/\mathbb{D}_n \arrow[r] & \dots .
\end{tikzcd} \tag{8.7}$$

The maps from $\mathbb{S}^1 \times (0,1]$ are the compositions of the arrows in the top row of diagram (8.6).

The Lipschitz map a is guaranteed to exist by the universal property of inverse limits since, by Corollary 8.0.16, $\mathbb{H}^1$ is the inverse limit of the inverse system given by the bottom row of diagram (8.7). The Lipschitz map b exists by the universal property of inverse limits since, by Lemma 8.0.19, $\mathbb{S}^1 \times [0,1]/\mathbb{S}^1 \times \{0\}$ is the inverse limit of the inverse system given by the top row of diagram (8.7).

Further, the composition $a \circ b$ must be equal to the map H since H is the unique map from $\mathbb{S}^1 \times (0,1]$ to $\mathbb{H}^1$ that allows diagram (8.1) to commute. Moreover, there is an equality of Lipschitz loops,

$$\gamma_0 = H(-,1) = a \circ b(-,1).$$

So, precomposing the Lipschitz map a by the Lipschitz map guaranteed in Observation 8.0.18 yields a Lipschitz extension of the loop γ_0. Thus, the homotopy class $[\gamma_0]$ is trivial in $\pi_1^{\mathrm{Lip}}(\mathbb{H}^1, 0)$ and the origin is non-singular. $\square$

Observation 8.0.21. Any point in a sub-Riemannian manifold that is complete with respect to the Carnot-Carathéodory metric is non-singular. The argument in Theorem 8.0.20 holds when $\mathbb{H}^1$ is exchanged for such a space. Recall that $\mathbb{H}^1$ as the inverse limit of an inverse system was a special case of Corollary 8.0.16.

Corollary 8.0.22. *Every point in a contact 3-manifold is non-singular.*

Proof. Let (M, ξ, g) be a contact 3-manifold with a base point $p \in M$. Let $[\gamma_0] \in \pi_1^{\mathrm{Lip}}((M, d_{CC}^M), p)$ be a homotopy class such that every neighborhood of p contains a representative of the class $[\gamma_0]$. In particular, there is a biLipschitz Darboux neighborhood of p by Corollary 6.0.11, that is, there exists a biLipschitz open embedding

$$\varphi : V \hookrightarrow (M, d_{CC}^M)$$

from an open neighborhood $V \subset \mathbb{H}^1$ of the origin such that $\varphi(0) = p$. By assumption, the homotopy class $[\gamma_0]$ has a representative in the open subset $\varphi(V)$. Thus, the homotopy class $[\gamma_0]$ is an element of $\pi_1^{\mathrm{Lip}}((\varphi(V), d_{CC}^M), p)$.

Then, $\varphi_{\#}^{-1}[\gamma_0]$ is a homotopy class based at the origin in $\mathbb{H}^1$. Furthermore, since φ is a biLipschitz embedding, the homotopy class $\varphi_{\#}^{-1}[\gamma_0]$ has a representative in every neighborhood of the origin in $\mathbb{H}^1$. By Theorem 8.0.20, $\varphi_{\#}^{-1}[\gamma_0]$ is the trivial class in $\mathbb{H}^1$ based at the origin. Therefore, $[\gamma_0]$ is the trivial class in M that is based at p.

□

Definition of universal path space

Fix a manifold with a metric (M, d) as well as a base point $p_0 \in M$. We will now construct the universal (Lipschitz) path space for the manifold (M, d). This

construction, as well as several of the insights and results that follow, are inspired by the work of Bogley and Sieraski in [4].

Definition 8.0.23. Two Lipschitz paths $\gamma, \gamma' : I \longrightarrow (M, d)$ are *homotopic rel endpoints*, denoted $\gamma \simeq \gamma'$, if the endpoints of the paths agree, $\gamma(0) = \gamma'(0)$ and $\gamma(1) = \gamma'(1)$, and there exists a Lipschitz homotopy between γ and γ' that fixes these endpoints. For a path γ, the class of all such paths homotopic rel endpoints to γ is denoted $[\gamma]_{\gamma(0)}^{\gamma(1)}$. When it does not cause unneeded confusion, the endpoints will be dropped from the notation.

Definition 8.0.24. The *universal (Lipschitz) path space* of a metric space (M, d) as a set is

$$\mathcal{P}_{(M,d)} := \{[\gamma]_{p_0}^{\gamma(1)} \ : \ \gamma : [0,1] \overset{\text{Lipschitz}}{\longrightarrow} (M, d), \gamma(0) = p_0\}.$$

The base point of $\mathcal{P}_{(M,d)}$ is $[p_0]$, the class of loops based at p_0 in M that are Lipschitz null-homotopic.

Convention 8.0.25. As we will only be discussing the universal Lipschitz path space of a metric space, we will refer to the space simply as the universal path space.

<u>Metric structure on the universal path space</u>

Given that the original space has a metric, there is a natural way to lift the metric to a pseudo-metric on the universal path space. In the following definition and for the remainder of this chapter, we will denote the reverse of the path γ by $\overline{\gamma}$.

Definition 8.0.26. The *lifted pseudo-metric* on the universal path space is given by

$$d_{\mathcal{P}}([\gamma], [\eta]) := \inf\{|\alpha(I)| \ : \ \alpha \simeq \overline{\gamma} * \eta\}$$

for $[\gamma], [\eta] \in \mathcal{P}_{(M,d)}$.

The lifted pseudo-metric is not in general a metric. The harmonic archipelago is an example of a metric space whose lifted pseudo-metric on its universal path space is not a metric. See Example 1.1 in [4]. Alas, for contact 3-manifolds, the lifted pseudo-metric is always a metric. This follows from all points of the space being non-singular.

Lemma 8.0.27. *For a contact 3-manifold with Carnot-Carathéodory metric structure* (M, d_{CC}^M)*, the lifted pseudo-metric* $d_{\mathcal{P}}$ *on the universal path space* $\mathcal{P}_{(M,d_{CC}^M)}$ *is a metric. In this case,* $d_{\mathcal{P}}$ *is called a* lifted metric.

Proof. We will only concern ourselves with showing that the lifted pseudo-metric $d_{\mathcal{P}}$ satisfies positive definiteness. Symmetry and triangle inequality are obvious.

First, suppose that $[f],[g] \in \mathcal{P}_{(M,d_{CC}^M)}$ are homotopy classes of Lipschitz paths based at the point p_0 such that the classes are equal, $[f] = [g]$. This is equivalent to the end points $f(1) = g(1)$ agreeing and the paths being Lipschitz homotopic for any representatives f and g selected. So, these paths can be concatenated to a loop $\overline{f} * g$ based at p_0 that is Lipschitz null-homotopic. Thus, the value $d_{\mathcal{P}}([f],[g]) = 0$ is zero.

Now, suppose for homotopy classes of paths $[f],[g] \in \mathcal{P}_{(M,d_{CC}^M)}$ that the lifted pseudo-metric evaluated on these two elements is zero: $d_{\mathcal{P}}([f],[g]) = 0$. We will take advantage of the infimum definition of $d_{\mathcal{P}}$ to show that the class of Lipschitz paths $\left[\overline{f} * g\right]$ based at $f(1)$ has a representative in arbitrarily small neighborhoods of $f(1)$. Corollary 8.0.22 will immediately imply that the class $\left[\overline{f} * g\right]$ is trivial and that f and g are Lipschitz homotopic rel endpoints.

For any positive value $\varepsilon > 0$, denote the open ball in M of radius ε centered at a point $p \in M$ by $B_{CC}^M(p,\varepsilon)$. If $\alpha : (I,0) \to (M,p)$ is a Lipschitz path at p whose diameter is less that ε, then the path is contained within the open ball $\alpha(I) \subset B_{CC}^M(p,\varepsilon)$.

Let the positive value $\varepsilon > 0$ be given. Since the distance between the classes $[f]$

and $[g]$ is zero, there exists a path

$$\alpha_\varepsilon : (I, 0) \longrightarrow (M, f(1))$$

such that α_ε is Lipschitz homotopic rel endpoints to the path $\overline{f} * g$ and the diameter of the path $|\alpha(I)| < \varepsilon$ is bounded by ε. Thus, the image of α_ε is contained in the open ball $B^M_{CC}(p, \varepsilon)$.

Finally, we will use that ε can be chosen to be arbitrarily small. Since, for any $\varepsilon > 0$, the image of α_ε is contained in the open ball of radius ε centered at $f(1)$, the endpoint $\alpha_\varepsilon(1) = g(1)$ is arbitrarily close to the point $f(1)$. Thus, the endpoints of f and g must agree and, for any $\varepsilon > 0$, the path α_ε is in fact a Lipschitz loop.

Now, for every neighborhood $B^M_{CC}(f(1), \varepsilon)$ of the point $f(1)$, the loop $\overline{f} * g$ has a representative contained within the open ball of radius ε. Since these open balls form a basis for the topology of M, every neighborhood of the point $f(1)$ has a loop homotopic to $\overline{f} * g$. By Corollary 8.0.22, since the point $f(1)$ is non-singular, the loop $\overline{f} * g$ is null-homotopic and the paths f and g are Lipschitz homotopic rel endpoints.

□

So, the universal path space of a contact 3-manifold is a metric space with the lifted metric. We will denote the open ball centered at $[\gamma] \in \mathcal{P}_{(M, d^M_{CC})}$ with radius $\varepsilon > 0$ by $B_{\mathcal{P}}([\gamma], \varepsilon)$.

Observation 8.0.28. The universal path space of any sub-Riemannian manifold that is complete with respect to the Carnot-Carathéodory metric is also a metric space with the lifted metric. The argument for Lemma 8.0.27 works in this situation since each point in such a sub-Riemannian manifold is non-singular (Observation 8.0.21).

Endpoint projection

Provided that the metric space M has a base point $p_0 \in M$, then there is a natural base point for $\mathcal{P}_{(M,d)}$ which is $[p_0]$, the class of Lipschitz loops in M based at p_0 which are Lipschitz homotopic to the constant loop.

Furthermore, there is a based map from the universal path space $\mathcal{P}_{(M,d)}$ to the original manifold M given by endpoint projection:

$$\begin{aligned} \pi : \quad \mathcal{P}_{(M,d)} &\to M \\ [\gamma]_{p_0}^{\gamma(1)} &\mapsto \gamma(1). \end{aligned}$$

The map π is obviously well-defined as all paths contained in the equivalence class $[\gamma]_{p_0}^{\gamma(1)}$ have the same endpoint $\gamma(1)$.

Lemma 8.0.29. *Let (M, ξ, g) be a sub-Riemannian manifold that is complete with respect to d_{CC}^M. The endpoint projection*

$$\pi : (\mathcal{P}_{(M,d)}, d_{\mathcal{P}}) \to (M, d_{CC}^M)$$

is a metric map.

Proof. First, we will show that the diameter of the image of a geodesic agrees with the length of the geodesic. Then, we will argue that the diameter of a geodesic between two points is the infimum of diameters of paths over all paths joining the points. Finally, we will show that this is enough to guarantee that endpoint projection is a metric map.

First, we will show that the diameter of the image of a geodesic agrees with the distance between the points it joins. Let p and p' be points in M. Since M is complete, as is articulated in [24], there exists a (length-minimizing) geodesic $\gamma_p^{p'} : I \to (M, d_{CC}^M)$ joining p and p'.

We will compute the diameter of the image of $\gamma_p^{p'}$. The diameter of $\mathrm{Im}(\gamma_p^{p'})$ is

$$|\mathrm{Im}(\gamma_p^{p'})| = \sup_{t \neq t'} d_{CC}^M \left(\gamma_p^{p'}(t), \gamma_p^{p'}(t')\right).$$

Since geodesics restrict to geodesics, for any t and t' in the domain of $\gamma_p^{p'}$, the Carnot-Carathéodory distance between $\gamma_p^{p'}(t)$ and $\gamma_p^{p'}(t')$ is determined by the length of the restriction of the geodesic:

$$d_{CC}^M \left(\gamma_p^{p'}(t), \gamma_p^{p'}(t')\right) = l^M \left(\gamma_p^{p'}|_{[t,t']}\right).$$

By Lemma 2.0.53, any restriction of a geodesic yields a shorter path. Thus, the diameter $|\mathrm{Im}(\gamma_p^{p'})|$ is bounded above by $l^M(\gamma_p^{p'})$. In fact, this is an equality as $t = 0$ and $t' = d_{CC}^M(p, p')$ yields the length of the geodesic. Thus, the diameter of the image of a geodesic is the distance between the points it joins:

$$|\mathrm{Im}(\gamma_p^{p'})| = l^M \left(\gamma_p^{p'}\right) = d_{CC}^M(p, p').$$

Let $\beta : I \to (M, \xi)$ be any other horizontal path joining the points p and p'. Then, the diameter of $\mathrm{Im}(\beta)$ is bounded below by the diameter of $\mathrm{Im}(\gamma_p^{p'})$:

$$|\mathrm{Im}(\beta)| = \sup_{t \neq t'} d_{CC}^M(\beta(t), \beta(t')) \geq d_{CC}^M(\beta(0), \beta(1)) = d_{CC}^M(p, p') = |\mathrm{Im}(\gamma_p^{p'})|.$$

Thus, the infimum of diameters of paths joining p and p' is $|\mathrm{Im}(\gamma_p^{p'})|$:

$$\inf\{|\mathrm{Im}(\beta)| \ : \ \beta \text{ joins } p \text{ and } p'\} = |\mathrm{Im}(\gamma_p^{p'})|$$

With these relationships expressed, we will show that endpoint projection is a metric map. Let $[\eta]_{p_0}^{\eta(1)}, [\alpha]_{p_0}^{\alpha(1)} \in \mathcal{P}_{(M,d)}$. Note that if a map β is homotopic rel

endpoints of the concatenation $\overline{\eta} * \alpha$, then β is a path that joins the endpoints of η and α. Then, we have the following string of inequalities:

$$\begin{aligned}
d_{\mathcal{P}}\left([\eta]_{p_0}^{\eta(1)}, [\alpha]_{p_0}^{\alpha(1)}\right) &= \inf\{|\operatorname{Im}(\beta)| \ : \ \beta \simeq \overline{\eta} * \alpha\} \\
&\geq \inf\{|\operatorname{Im}(\beta)| \ : \ \beta \text{ joins } \eta(1) \text{ and } \alpha(1)\} \\
&= |\operatorname{Im}(\gamma_{\eta(1)}^{\alpha(1)})| \\
&= d_{CC}^{M}(\eta(1), \alpha(1)) \\
&= d_{CC}^{M}(\pi[\eta]_{p_0}^{\eta(1)}, \pi[\alpha]_{p_0}^{\alpha(1)}).
\end{aligned}$$

Therefore, the map π is a metric map.

$\square$

Fibers of endpoint projection

In this subsection, we will show that the fiber over a point in a contact 3-manifold with respect to endpoint projection is totally disconnected and perfect.

Fix a contact 3-manifold (M, ξ, g) and its associated Carnot-Carathéodory metric d_{CC}^{M}. By Lemma 8.0.27, the universal path space $(\mathcal{P}_{(M, d_{CC}^{M})}, d_{\mathcal{P}})$ is a metric space. For any point $z \in M$, the fiber with respect to endpoint projection

$$\pi^{-1}(z) = \{[\gamma] \in \mathcal{P}_{(M, d_{CC}^{M})} \ : \ \pi[\gamma] = \gamma(1) = z\}$$

is a subset of $\mathcal{P}_{(M, d_{CC}^{M})}$. It is endowed with the subspace topology with respect to the metric topology on the universal path space. Thus, the basis for the topology on $\pi^{-1}(z)$ is the collection

$$\{B_{\mathcal{P}}([\gamma], \varepsilon) \cap \pi^{-1}(z) \ : \ [\gamma] \in \mathcal{P}_{(M, d_{CC}^{M})}, \varepsilon > 0\}.$$

We will now define a topology on the first Lipschitz homotopy group of (M, d_{CC}^M) based at the point z called the *local subgroup topology*. We will then show that any element of the fiber over z yields a homeomorphism between $\pi^{-1}(z)$ and $\pi_1^{\mathrm{Lip}}(M, z)$.

Definition 8.0.30. Let (M, d) be a metric space. For a point $z \in M$ and an open neighborhood U of z, the *local subgroup* of U at z is the image of the group $\pi_1^{\mathrm{Lip}}(U, z)$ in the group $\pi_1^{\mathrm{Lip}}(M, z)$ under the homomorphism induced by inclusion:

$$\Pi_M^U(z) := \mathrm{Im}\left(\pi_1^{\mathrm{Lip}}(U, z) \longrightarrow \pi_1^{\mathrm{Lip}}(M, z)\right).$$

Let $\Sigma(z)$ denote the set of all local subgroups $\Pi_M^U(z)$ for all open neighborhood $U \subset M$ of the point z.

Observation 8.0.31. A local subgroup $\Pi_M^U(z)$ has a right action on the group $\pi_1^{\mathrm{Lip}}(M, z)$ given by concatentation:

$$\begin{array}{ccc} \pi_1^{\mathrm{Lip}}(M, z) \times \Pi_M^U(z) & \to & \pi_1^{\mathrm{Lip}}(M, z) \\ ([\gamma], [\epsilon]) & \mapsto & [\gamma] \cdot [\epsilon] := [\gamma * \epsilon]. \end{array}$$

Definition 8.0.32. For a metric space (M, d) and a point $z \in M$, the *local subgroup topology* on the subgroup $\pi_1^{\mathrm{Lip}}(M, z)$ is the topology generated by the basis

$$\pi_1^{\mathrm{Lip}}(M, z)/\Sigma(z) = \{[\gamma] \cdot \Pi_M^U(z) \;:\; [\gamma] \in \pi_1^{\mathrm{Lip}}(M, z), z \in U \overset{\text{open}}{\subset} M\},$$

the set of all cosets in $\pi_1^{\mathrm{Lip}}(M, z)$ of all local subgroups $\Pi_M^U(z) \in \Sigma(z)$.

Lemma 8.0.33. *Take a metric space (M, d) and a point $z \in M$. The collection of cosets $\pi_1^{Lip}(M, z)/\Sigma(z)$ forms a basis for a topology on the group $\pi_1^{Lip}(M, d)$.*

Proof. Let U and V be open neighborhoods of z in M and let $[g], [h] \in \pi_1^{\mathrm{Lip}}(M, z)$. The cosets $[g] \cdot \Pi_M^U(z)$ and $[h] \cdot \Pi_M^V(z)$ are elements of the set $\pi_1^{\mathrm{Lip}}(M, z)/\Sigma(z)$.

Let $[f]$ be an element of the intersection of the two cosets: $[f] \in [g] \cdot \Pi_M^U(z) \cap [h] \cdot \Pi_M^V(z)$. Consider the coset

$$[f] \cdot \Pi_M^{U \cap V}(z).$$

This coset contains the element $[f]$ since the constant loop $[z]$ is an element of the local subgroup $\Pi_M^{U \cap V}(z)$.

An element of the coset $[f] \cdot \Pi_M^{U \cap V}(z)$ can be represented by $[f][\delta]$ where $[\delta]$ has a representative loop δ contained in the open subset $U \cap V$. Since the element $[f]$ is assumed to be an element of the coset $[g] \cdot \Pi_M^U(z)$, there exists an equivalence class $[\epsilon] \in \Pi_M^U(z)$ which has a representative loop ϵ contained in the open subset U such that $[f] = [g] \cdot [\epsilon]$. Thus, we have an equality of equivalence classes:

$$[f] \cdot [\delta] = [g] \cdot [\epsilon] \cdot [\delta] = [g] \cdot [\epsilon * \delta].$$

The equivalence class $[\epsilon * \delta]$ is contained in $\Pi_M^U(z)$ since the loop ϵ is contained in U and the loop δ is contained in $U \cap V \subset U$. Therefore, $[f][\delta] \in [g] \cdot \Pi_M^U(z)$.

A similar argument shows that $[f][\delta] \in [h] \cdot \Pi_M^V(z)$. Thus,

$$[f] \cdot \Pi_M^{U \cap V}(z) \subset [g] \cdot \Pi_M^U(z) \cap [h] \cdot \Pi_M^V(z).$$

This completes the proof. $\square$

As we proceed, we will assume that $\pi_1^{\mathrm{Lip}}(M, z)$ has the local subgroup topology.

Observation 8.0.34. For a point $z \in M$ in a metric space (M, d) and an element $[f] \in \pi^{-1}(z)$ in the fiber over z, concatenation defines a map from the first Lipschitz homotopy group based at z to the fiber over z:

$$\begin{array}{rccc} \phi_{[f]}: & \pi_1^{\mathrm{Lip}}(M,z) & \to & \pi^{-1}(z) \\ & [\gamma] & \mapsto & [f]\cdot[\gamma] := [f * \gamma]. \end{array}$$

Lemma 8.0.35. *Let (M,d) be a based metric space such that the universal path space $\mathcal{P}_{(M,d)}$ is a metric space. Let $z \in M$ and $[f] \in \pi^{-1}(z)$ be an element of the fiber over z. Then, the map*

$$\phi_{[f]} : \pi_1^{Lip}(M,z) \longrightarrow \pi^{-1}(z)$$

is a homeomorphism.

Proof. We begin by arguing that the map $\phi_{[f]}$ is a bijection. Let $[\alpha],[\alpha'] \in \pi_1^{\mathrm{Lip}}(M,z)$ such that $[f * \alpha] = [f * \alpha']$. The reverse of f, denoted $\overline{f}$, is such that $\overline{f} * f$ is Lipschitz null homotopic. Thus, we have equality of equivalence classes

$$[\alpha] = [\overline{f} * f * \alpha] = [\overline{f} * f * \alpha'] = [\alpha'].$$

Therefore, the map $\phi_{[f]}$ is injective.

Let $[h] \in \pi^{-1}(z)$ be an element of the fiber over z, that is, $[h]$ is a equivalence class of paths from the base point z_0 to the point z. As the element $[f]$ also represents such an equivalence class, we can concatenate $[\overline{f}]$ and $[h]$ to obtain an element $[\overline{f} * h]$ in $\pi_1^{\mathrm{Lip}}(M,d)$. Then, the image of this element by the map $\phi_{[f]}$ is $[h]$:

$$\phi_{[f]}([\overline{f} * h]) = [f * \overline{f} * h] = [h].$$

Thus, the map $\phi_{[f]}$ is surjective.

To see that the map is a homeomorphism, we will note that the map $\phi_{[f]}$ maps basis elements of the local subgroup topology to basis elements of the subspace topology. Indeed, let $[g]\cdot\Pi_M^{B(z,\varepsilon)}(z)$ be a basis element of the local subgroup topology on $\pi_1^{\mathrm{Lip}}(M,z)$. Then, the image of this basis element is

$$\begin{aligned}\phi_{[f]}\left([g]\cdot\Pi_M^{B(z,\varepsilon)}(z)\right) &= \{[f*g]\cdot[\delta]\ :\ [\delta]\in\pi_1^{\mathrm{Lip}}(B(z,\varepsilon),z)\}\\ &= B_{\mathcal{P}}\left([f*g],2\varepsilon\right)\cap\pi^{-1}(z),\end{aligned}$$

which is a basis element for $\pi^{-1}(z)$ with the subspace topology. □

With these spaces established as homeomorphic, we will use a result of [4] to establish some facts about the topology of the fibers of the endpoint projection.

Theorem 8.0.36. *Let (M, d_{CC}^M) be a contact 3-manifold. For any $z\in M$, the fiber $\pi^{-1}(z)$ with the local subgroup topology is totally disconnected and perfect.*

Proof. By Corollary 8.0.22, the point $z\in M$ is non-singular. Thus, the infinitesimal subgroup $\cap\{S\ :\ S\in\Sigma(z)\}$ is the trivial subgroup. As is stated in Theorem 2.9(c) in [4], the infinitesimal subgroup being trivial is equivalent to the group $\pi_1^{\mathrm{Lip}}(M,z)$ with the local subgroup topology being totally disconnected.

By Lemma 8.0.2, the point z is not a regular point. With the new language introduced in this section, z not being regular is equivalent to the trivial group not being included in $\Sigma(z)$. Theorem 2.9(d) in [4] states that a group with the local subgroup topology is discrete if and only if the trivial group is in $\Sigma(z)$. Thus, the group $\pi_1^{\mathrm{Lip}}(M,z)$ is not discrete. Theorem 2.9(e) in [4] states that a group that is totally disconnected and not discrete is perfect, i.e., every element of the group is an accumulation point for the group. Thus, $\pi_1^{\mathrm{Lip}}(M,z)$ is a perfect topological space with the local subgroup topology.

By Lemma 8.0.35, the fiber $\pi^{-1}(z)$ is totally disconnected and perfect.

□

Lifts of paths to the universal path space

Now, we show that there exists Lipschitz lifts of Lipschitz paths in M. Let $\gamma : (I,0) \to (M,p_0)$ be a Lipschitz path. For any time $t \in I$, there exists a reparametrization of the path γ such that the endpoint of the reparametrization is the point $\gamma(t)$:

$$\begin{aligned} \gamma_t : \quad (I,0) &\to (M,p_0) \\ s &\mapsto \gamma(ts). \end{aligned}$$

It is clear that this path is also Lipschitz. The path γ_t then determines an element of the universal path space, $[\gamma_t]_{p_0}^{\gamma(t)} \in \mathcal{P}_{(M,d)}$. Thus, there is a path in the universal path space:

$$\begin{aligned} \widehat{\gamma} : \quad (I,0) &\to (\mathcal{P}_{(M,d)}, [p_0]) \\ t &\mapsto [\gamma_t]_{p_0}^{\gamma(t)}. \end{aligned}$$

This is a lift of the original path γ as endpoint projection returns $\gamma(t)$ for each time $t \in I$:

$$\begin{tikzcd} & \mathcal{P}_{(M,d)} \arrow[d, "\pi"] \\ I \arrow[ur, dashed, "\widehat{\gamma}"] \arrow[r, "\gamma"'] & M. \end{tikzcd}$$

As the next lemma shows, this lift is indeed Lipschitz.

Lemma 8.0.37. *Let $\gamma : (I,0) \longrightarrow (M,p_0)$ be an L-Lipschitz path. Then, the lifted path $\widehat{\gamma} : (I,0) \longrightarrow (\mathcal{P}_{(M,d)}, [p_0])$ is Lipschitz with respect to the lifted metric.*

Proof. Let $t,t' \in I$. The following string of inequalities follows from the path $\gamma_{[t,t']}$ (properly reparamterized) being an element over which the infimum is being taken

over and Lemma 2.0.68:

$$\begin{aligned} d_{\mathcal{P}}(\widehat{\gamma}(t), \widehat{\gamma}(t')) &= \inf\{|\alpha(I)| \ : \ \alpha \simeq \overline{\gamma_t} * \gamma_{t'}\} \\ &= |\gamma_{[t,t']}(I)| \\ &\leq L \cdot |[t,t']| \\ &= L \cdot |t' - t|. \end{aligned}$$

□

We now extend this idea to Lipschitz maps of metric trees into (M, d). Metric trees have unique geodesics between points and we already know that we can lift paths to paths in the universal path space, so we combine these two ideas.

Let (T, d^T) be a metric tree with a base point $x_0 \in T$ and suppose we have a based, Lipschitz map

$$f : (T, x_0) \longrightarrow (M, p_0).$$

Let $x \in T$ be any point in the metric tree. Then, there exists a unique geodesic, i.e., length-minimizing path, between the points x_0 and x,

$$\beta_x : (I, 0, 1) \longrightarrow (T, x_0, x).$$

Note that this map is better than Lipschitz, it is an isometry. Then, the composition $f \circ \beta_x : I \to M$ is a Lipschitz path in M such that $f \circ \beta_x(0) = p_0$ and $f \circ \beta_x(1) = f(x)$. Then, we define a map of sets:

$$\begin{aligned} \widehat{f} : \ T &\to \mathcal{P}_{(M,d)} \\ x &\mapsto [f \circ \beta_x]. \end{aligned}$$

Note that for the base point $x_0 \in T$, that the unique geodesic from x_0 to itself is

constant. Thus, the composition $f \circ \beta_{x_0}$ is constant and, since the map f is based,

$$[f \circ \beta_{x_0}] = [p_0]$$

and the map $\widehat{f}$ is based. Also, the map $\widehat{f}$ is indeed a lift of the map f:

$$\begin{tikzcd} & (\mathcal{P}_{(M,d)}, [p_0]) \arrow[d, "\pi"] \\ (T, x_0) \arrow[ur, dashed, "\widehat{f}"] \arrow[r, "f"'] & (M, p_0). \end{tikzcd}$$

Lemma 8.0.38. *Let $f : (T, x_0) \longrightarrow (M, p_0)$ be a Lipschitz, based map from a metric tree (T, d^T). Then, the lifted map $\widehat{f} : (T, x_0) \longrightarrow (\mathcal{P}_{(M,d)}, [p_0])$ is Lipschitz with respect to the lifted metric.*

Proof. Let $x, x' \in T$. By definition of the metric on the universal path space, there is an equality of values

$$d_{\mathcal{P}}(\widehat{f}(x), \widehat{f}(x')) \;\; = \;\; \inf\{|\alpha(I)| \; : \; \alpha \simeq \overline{f \circ \beta_x} * f \circ \beta_{x'}\}.$$

Since the path

$$\overline{f \circ \beta_x} * f \circ \beta_{x'} = f \circ (\overline{\beta_x} * \beta_{x'})$$

is among the paths the infimum is being taken over, we have the inequality

$$d_{\mathcal{P}}(\widehat{f}(x), \widehat{f}(x')) \;\; \leq \;\; |f \circ (\overline{\beta_x} * \beta_{x'})(I)|.$$

Let L be the Lipschitz constant associated to the Lipschitz map f. Then, we have

the string of inequalities:

$$\begin{aligned} d_{\mathcal{P}}(\widehat{f}(x), \widehat{f}(x')) &\leq |f \circ (\overline{\beta_x} * \beta_{x'})(I)| \\ &\leq L \cdot |\overline{\beta_x} * \beta_{x'}(I)| \\ &= L \cdot d^T(x, x'). \end{aligned}$$

This last equality comes from the concatenation of paths $\overline{\beta_x} * \beta_{x'}$ being the unique geodesic in T connecting the points x and x' and that diameter of a geodesic is the same as the distance between its endpoints in a metric tree.

□

Universal path space of a contact 3-manifold is a metric tree

We now return to the case where the space M is a contact 3-manifold. Recall that (M, d^M_{CC}) is purely 2-unrectifiable and, via Theorem 5 in [34], any Lipschitz map from a quasi-convex, Lipschitz simply connected space factors through a metric tree. An immediate corollary of Theorem 5 and Lemma 8.0.38 is that any Lipschitz map from a quasi-convex, Lipschitz simply connected metric space to a purely 2-unrectifiable space lifts to a Lipschitz map into the universal path space of the target.

Corollary 8.0.39. *Let (X, d^X) be a quasi-convex, Lipschitz simply-connected metric space and let (M, d) be a purely 2-unrectifiable space. Then, for any Lipschitz map $f : (X, d^X) \longrightarrow (M, d)$, there is a Lipschitz lift $\widehat{f} : (X, d^X) \longrightarrow (\mathcal{P}_{(M,d)}, d_{\mathcal{P}})$ such that the map $\widehat{f}$ factors through a metric tree and the following diagram commutes:*

$$\begin{tikzcd} & \mathcal{P}_{(M,d)} \arrow[d, "\pi"] \\ X \arrow[ur, dashed, "\widehat{f}", "\exists"'] \arrow[r, "f"'] & M. \end{tikzcd}$$

Proof. By Theorem 5 in [34], the Lipschitz map f factors through a metric tree (T, d^T):

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\psi"'] & & M, \\ & T \arrow[ur, "\varphi"'] & \end{tikzcd}$$

where the maps ψ and φ are both Lipschitz. By Lemma 8.0.38, there is a lift of the Lipschitz map φ to a Lipschitz map $\widehat{\varphi} : T \to \mathcal{P}_{(M,d)}$. Thus, we have the following commutative diagram,

$$\begin{tikzcd} & & \mathcal{P}_{(M,d)} \arrow[d, "\pi"] \\ X \arrow[urr, dotted, "\widehat{\varphi}\circ\psi", "\exists"'] \arrow[rr, "f"'] \arrow[dr, "\psi"'] & & M, \\ & T \arrow[ur, "\varphi"'] \arrow[uur, dashed, "\widehat{\varphi}"] & \end{tikzcd}$$

and the map $\widehat{f} = \widehat{\varphi} \circ \psi$ is the desired lift.

$\square$

Corollary 8.0.40. *For a purely 2-unrectifiable space (M, d) with base point p_0, the map of first Lipschitz homotopy groups induced by the Lipschitz map π:*

$$\pi_{\#} : \pi_1^{Lip}((\mathcal{P}_{(M,d)}, d_{\mathcal{P}}), [p_0]) \longrightarrow \pi_1^{Lip}((M, d), p_0)$$

is injective.

Proof. Let equivalence class $[F] \in \pi_1^{\mathrm{Lip}}((\mathcal{P}_{(M,d)}, d_{\mathcal{P}}), [p_0])$ be in the kernel of the homomorphism $\pi_{\#}$. Thus, the Lipschitz loop $\pi \circ F$ in M is null-homotopic. So, there exists a Lipschitz homotopy $H : \mathbb{D}^2 \to M$ such that the following diagram commutes.

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r, "F"] \arrow[d, hook] & \mathcal{P}_{(M,d)} \arrow[d, "\pi"] \\ \mathbb{D}^2 \arrow[r, "H"] \arrow[ur, dashed, "\widehat{H}", "\exists"'] & M. \end{tikzcd}$$

By Corollary 8.0.39, there exists a Lipschitz lift of the map H to a map $\widehat{H}$ as indicated in the diagram. Thus, the loop F in the universal path space is null-homotopic and $[F] = [p_0]$. Therefore, the kernel of the homomorphism $\pi_\#$ is trivial.

$\square$

Lemma 8.0.41. *Let (M,d) be a purely 2-unrectifiable metric space. Assume that the universal path space has the unique path lifting property. Then, the universal path space $\mathcal{P}_{(M,d)}$ is Lipschitz simply-connected*

Proof. By Corollary 8.0.40, the homomorphism $\pi_\# : \pi_1^{\mathrm{Lip}}((\mathcal{P}_{(M,d)}, d_{\mathcal{P}}), [p_0]) \longrightarrow \pi_1^{\mathrm{Lip}}((M,d), p_0)$ is injective. Thus, in order to show that the group $\pi_1^{\mathrm{Lip}}((\mathcal{P}_{(M,d)}, d_{\mathcal{P}}), [p_0])$ is trivial, it is enough to show that the image of $\pi_\#$ is trivial in the group $\pi_1^{\mathrm{Lip}}((M,d), p_0)$.

Let $[\gamma]$ be in the image of $\pi_\#$. So, there exists representative $\gamma : (I, \{0,1\}) \to (M, p_0)$ in $[\gamma]$ such that the loop γ is the image of a loop in $\mathcal{P}_{(M,d)}$. By the unique lifting property, the loop in $\mathcal{P}_{(M,d)}$ must be $\widehat{\gamma}$. Since $\widehat{\gamma}$ is a loop, there is an equality of the initial and terminal points of the path $\widehat{\gamma}$,

$$[p_0] = \widehat{\gamma}(0) = \widehat{\gamma}(1) = [\gamma].$$

Thus, γ is Lipschitz null-homotopic and the image of $\pi_\#$ is the identity element $[p_0]$ in $\pi_1^{\mathrm{Lip}}((M,d), p_0)$.

$\square$

Observation 8.0.42. Let (X, d^X) be a quasi-convex, Lipschitz simply-connected space. Let (Y, d^Y) be a metric space with the following property:

$$\text{For any Lipschitz map } \varphi : \mathbb{D}^2 \longrightarrow (Y, d^Y),\ \mathcal{H}^2(\mathrm{Im}(\varphi)) = 0. \tag{P}$$

Then, any Lipschitz map $f : X \longrightarrow Y$ factors through a metric tree. This follows from the proof of Theorem 5 by Wenger and Young in [34]. The full power of the target metric space being purely 2-unrectifiable is not used in the proof, just property (P). As such, the result can be weakened from purely 2-unrectifiable to the property (P) concerning Lipschitz maps out of the 2-disk.

We now note that the universal path space of a purely 2-unrectifiable space has property (P) if the universal path space has the unique lifting property.

Lemma 8.0.43. *Let (M, d) be a purely 2-unrectifiable metric space with universal path space $(\mathcal{P}_{(M,d)}, d_{\mathcal{P}})$. Assume that $\pi : \mathcal{P}_{(M,d)} \longrightarrow M$ has the unique lifting property. Then, the metric space $(\mathcal{P}_{(M,d)}, d_{\mathcal{P}})$ has property (P).*

Proof. Let $F : \mathbb{D}^2 \longrightarrow \mathcal{P}_{(M,d)}$ be a Lipschitz map. Then, the composition $\pi \circ F$ is a Lipschitz map from the 2-disk to the purely 2-unrectifiable space (M, d). By Corollary 8.0.39, there is a Lipschitz lift of the map $\pi \circ F$:

$$\begin{tikzcd} & \mathcal{P}_{(M,d)} \arrow[d, "\pi"] \\ \mathbb{D}^2 \arrow[ur, bend right, "F"'] \arrow[ur, dotted, bend left, "\widehat{\pi\circ F}"] \arrow[r, "\pi\circ F"'] & M. \end{tikzcd}$$

By the unique lifting property, there is an equality of Lipschitz maps $F = \widehat{\pi \circ F}$.

Also, from Corollary 8.0.39, the Lipschitz map F factors through a metric tree,

$$\begin{tikzcd} \mathbb{D}^2 \arrow[rr, "F"] \arrow[dr, "\psi"'] & & \mathcal{P}_{(M,d)}, \\ & T \arrow[ur, "\widehat{\varphi}"'] & \end{tikzcd}$$

where the maps ψ and $\widehat{\varphi}$ are Lipschitz. Since metric trees are purely 2-unrectifiable, and by Lemma 2.0.68, the 2-dimensional Hausdorff measure of the image of F is 0:

$$\mathcal{H}^2(\mathrm{Im}(F)) = \mathcal{H}^2(\mathrm{Im}(\widehat{\varphi} \circ \psi)) \leq L \cdot \mathcal{H}^2(\mathrm{Im}(\psi)) = 0,$$

where L is the Lipschitz constant associated to the map $\widehat{\varphi}$. Thus, the metric space $(\mathcal{P}_{(M,d)}, d_{\mathcal{P}})$ has property (P).

□

Thus, we see that, in order to show that the universal path space is a metric tree, we only need to verify several properties.

Lemma 8.0.44. *For a purely 2-unrectifiable space (M, d), if the universal path space $\mathcal{P}_{(M,d)}$ is quasi-convex and has the unique lifting property, then the universal path space is a metric tree.*

Proof. Consider the identity map $\mathbb{1} : \mathcal{P}_{(M,d)} \longrightarrow \mathcal{P}_{(M,d)}$. Obviously, the map is Lipschitz. The domain is assumed to be quasi-convex. By and Lemma 8.0.41, the universal path space is also Lipschitz simply-connected. Since the universal path space has the unique lifting property, by Lemma 8.0.43, the target of the identity map $\mathcal{P}_{(M,d)}$ has property (P). By Observation 8.0.42, the identity map factors through a metric tree:

$$\begin{tikzcd} \mathcal{P}_{(M,d)} \arrow[rr, "\mathbb{1}"] \arrow[dr, "\psi"'] & & \mathcal{P}_{(M,d)}. \\ & T \arrow[ur, "\varphi"'] & \end{tikzcd}$$

By Theorem 5 in [34], the Lipschitz map ψ is surjective, and the Lipschitz map φ is its inverse. Thus, the universal path space is isomorphic to a metric tree.

$\square$

Thus, to attain Claim 8.0.0.2, it is enough to verify the following conjecture:

Conjecture 8.0.45. *For any purely 2-unrectifiable space (M,d), the universal path space $\mathcal{P}_{(M,d)}$ is quasi-convex and has the unique lifting property.*

Unique path lifting

Theorem 8.0.46. *Let (M,d) be a purely 2-unrectifiable metric space. The universal path space $(\mathcal{P}_{(M,d)}, d_{\mathcal{P}})$ has the unique path lifting property, that is, for any Lipschitz path $\gamma : (I,0) \to (M,p_0)$ there exists a unique Lipschitz path $\widehat{\gamma} : (I,0) \to (\mathcal{P}_{(M,d)}, [p_0])$ such that the following diagram commutes:*

$$\begin{tikzcd} & \mathcal{P}_{(M,d)} \arrow[d, "\pi"] \\ I \arrow[ur, dashed, "\widehat{\gamma}", "\exists!"'] \arrow[r, "\gamma"'] & M. \end{tikzcd}$$

Proof. The lift $\widehat{\gamma}$ was defined earlier in this chapter and was shown to be Lipschitz in Lemma 8.0.37. Let $\Gamma : (I,0) \to (\mathcal{P}_{(M,d)}, [p_0])$ be another Lipschitz lift of the path γ with Lipschitz constant L. It is enough to show that γ is a representative for the class of paths $\Gamma(1)$, that is, $\Gamma(1) = [\gamma] = \widehat{\gamma}(1)$. Any other $t \in I$ is a reparametrization of this case.

We begin by constructing, for each $N \in \mathbb{N}$, a representative η_N of $\Gamma(1)$ that is never more than $1/2^{N-1}$ away from the original path γ.

Let $N \in \mathbb{N}$. Partition the interval I by

$$0, \frac{1}{\lceil L \rceil 2^N}, \frac{2}{\lceil L \rceil 2^N}, \ldots, \frac{k}{\lceil L \rceil 2^N}, \frac{k+1}{\lceil L \rceil 2^N}, \ldots \frac{\lceil L \rceil 2^N - 1}{\lceil L \rceil 2^N}, 1,$$

where $\lceil L \rceil \in \mathbb{N}$ is the ceiling of L and $k \in \{0, 1, \ldots, \lceil L \rceil 2^N\}$. Also, fix a sequence of positive values $\varepsilon_N = \frac{1}{2^{N+1}}$.

We will proceed by induction to show that there exists a representative $\eta_N \in \Gamma(1)$ such that the path η_N restricted to the subinterval $\left[\frac{k}{\lceil L \rceil 2^N}, \frac{k+1}{\lceil L \rceil 2^N}\right]$ is mapped into the open ball

$$B\left(\gamma\left(\frac{k}{\lceil L \rceil 2^N}\right), \frac{1}{2^{N-1}} + 2\varepsilon_N\right).$$

Consider the interval $\left[0, \frac{1}{\lceil L \rceil 2^N}\right]$. Since Γ is L-Lipschitz, we have the following inequality:

$$d_{\mathcal{P}}\left(\Gamma(0), \Gamma\left(\frac{1}{\lceil L \rceil 2^N}\right)\right) \le \left(\frac{L}{\lceil L \rceil 2^N}\right) \le \frac{1}{2^N}.$$

Recalling the definition of the metric $d_{\mathcal{P}}$ and noting that $\Gamma(0) = [p_0]$, we have that

$$d_{\mathcal{P}}\left([p_0], \Gamma\left(\frac{1}{\lceil L \rceil 2^N}\right)\right) = \inf\left\{|\eta(I)| \ : \ \eta \in \Gamma\left(\frac{1}{\lceil L \rceil 2^N}\right)\right\} \le \frac{1}{2^N}.$$

Thus, there exists a representative $\eta_0^{1/\lceil L \rceil 2^N} \in \Gamma\left(\frac{1}{\lceil L \rceil 2^N}\right)$ that has diameter less than $\frac{1}{2^N} + \varepsilon_N$. The representative $\eta_0^{1/\lceil L \rceil 2^N}$ is a path from $\gamma(0) = p_0$ to $\gamma\left(\frac{1}{\lceil L \rceil 2^N}\right)$. Since we have an upper bound for the diameter of this path, we know that the image of $\eta_0^{1/\lceil L \rceil 2^N}$ is contained in an open ball centered at $\gamma(0) = p_0$,

$$\mathrm{Im}(\eta_0^{1/\lceil L \rceil 2^N}) \subset B\left(\gamma(0), \frac{1}{2^{N-1}} + 2\varepsilon_N\right).$$

Now, suppose that a representative $\eta_0^{k/\lceil L \rceil 2^N} \in \Gamma\left(\frac{k}{\lceil L \rceil 2^N}\right)$ exists such that, for $i \in \{0, \ldots, k-1\}$, the path $\eta_0^{k/\lceil L \rceil 2^N}$ restricted to the subinterval $\left[\frac{i}{\lceil L \rceil 2^N}, \frac{i+1}{\lceil L \rceil 2^N}\right]$ is contained in the open ball

$$B\left(\gamma\left(\frac{i}{\lceil L \rceil 2^N}\right), \frac{1}{2^{N-1}} + 2\varepsilon_N\right).$$

Consider the interval $\left[\frac{k}{\lceil L \rceil 2^N}, \frac{k+1}{\lceil L \rceil 2^N}\right]$. Again, since Γ is L-Lipschitz, we have the

inequality:

$$d_{\mathcal{P}}\left(\Gamma\left(\frac{k}{\lceil L\rceil 2^N}\right),\Gamma\left(\frac{k+1}{\lceil L\rceil 2^N}\right)\right)\le L\left(\frac{k+1}{\lceil L\rceil 2^N}-\frac{k}{\lceil L\rceil 2^N}\right)=\left(\frac{L}{\lceil L\rceil 2^N}\right)\le\frac{1}{2^N}.$$

Via the definition of $d_{\mathcal{P}}$, we have the inequality

$$\begin{aligned}
d_{\mathcal{P}}\left(\Gamma\left(\frac{k}{\lceil L\rceil 2^N}\right),\Gamma\left(\frac{k+1}{\lceil L\rceil 2^N}\right)\right) &= \inf\left\{|\eta(I)| \;:\; \eta\in\overline{\Gamma\left(\frac{k}{\lceil L\rceil 2^N}\right)}*\Gamma\left(\frac{k+1}{\lceil L\rceil 2^N}\right)\right\}\\
&= \inf\left\{|\eta(I)| \;:\; \eta\in\overline{\eta_0^{k/\lceil L\rceil 2^N}}*\Gamma\left(\frac{k+1}{\lceil L\rceil 2^N}\right)\right\}\\
&= \inf\left\{|\eta(I)| \;:\; \eta_0^{k/\lceil L\rceil 2^N}*\eta\in\Gamma\left(\frac{k+1}{\lceil L\rceil 2^N}\right)\right\}\\
&\le \frac{1}{2^N}.
\end{aligned}$$

Thus, there exists a path $\eta_{k/\lceil L\rceil 2^N}^{k+1/\lceil L\rceil 2^N}$ from $\eta_0^{k/\lceil L\rceil 2^N}\left(\frac{k}{\lceil L\rceil 2^N}\right)=\gamma\left(\frac{k}{\lceil L\rceil 2^N}\right)$ to $\gamma\left(\frac{k+1}{\lceil L\rceil 2^N}\right)$ that has diameter less than $\frac{1}{2^N}+\varepsilon_N$. Also, the following concatenation is a representative

$$\eta_0^{k/\lceil L\rceil 2^N}*\eta_{k/\lceil L\rceil 2^N}^{k+1/\lceil L\rceil 2^N}\in\Gamma\left(\frac{k+1}{\lceil L\rceil 2^N}\right).$$

Since we have an upper bound for the diameter of the path $\eta_{k/\lceil L\rceil 2^N}^{k+1/\lceil L\rceil 2^N}$, we know that the image of $\eta_{k/\lceil L\rceil 2^N}^{k+1/\lceil L\rceil 2^N}$ is contained in an open ball centered at $\gamma\left(\frac{k}{\lceil L\rceil 2^N}\right)$,

$$\mathrm{Im}(\eta_{k/\lceil L\rceil 2^N}^{k+1/\lceil L\rceil 2^N})\subset B\left(\gamma\left(\frac{k}{\lceil L\rceil 2^N}\right),\frac{1}{2^{N-1}}+2\varepsilon_N\right).$$

Thus, by induction, a representative $\eta_N\in\Gamma(1)$ exists such that

$$\mathrm{Im}\left(\eta_N\Big|_{\left[\frac{k}{\lceil L\rceil 2^N},\frac{k+1}{\lceil L\rceil 2^N}\right]}\right)\subset B\left(\gamma\left(\frac{k}{\lceil L\rceil 2^N}\right),\frac{1}{2^{N-1}}+2\varepsilon_N\right).$$

The path η_N can be written as the concatenation of these restrictions

$$\eta_N = \eta_0^{1/\lceil L\rceil 2^N} * \eta_{1/\lceil L\rceil 2^N}^{2/\lceil L\rceil 2^N} * \ldots * \eta_{\lceil L\rceil 2^N - 1/\lceil L\rceil 2^N}^{1}. \tag{8.8}$$

Now, each η_N is in $\Gamma(1)$. Thus, for any $N, N' \in \mathbb{N}$, the paths η_N and $\eta_{N'}$ are homotopic. In fact, as will be argued, for each $N \in \mathbb{N}$ and each $k \in \{0, 1, \ldots, \lceil L\rceil 2^N\}$, the restricted paths

$$\eta_{k/\lceil L\rceil 2^N}^{k+1/\lceil L\rceil 2^N} \simeq \eta_{k/\lceil L\rceil 2^N}^{2k+1/\lceil L\rceil 2^{N+1}} * \eta_{2k+1/\lceil L\rceil 2^{N+1}}^{k+1/\lceil L\rceil 2^N} \tag{8.9}$$

are homotopic. By construction, for each $N \in \mathbb{N}$ and each $k \in \{0, 1, \ldots, \lceil L\rceil 2^N - 1\}$, the restricted paths

$$\eta_N \Big|_{\left[0, \frac{k}{\lceil L\rceil 2^N}\right]} \simeq \eta_{N+1} \Big|_{\left[0, \frac{k}{\lceil L\rceil 2^N}\right]}$$

are homotopic. Each can be written as a concatenation of paths, per (8.8). Thus, we have the following desired equivalences:

$$\begin{aligned}
\eta_{k/\lceil L\rceil 2^N}^{k+1/\lceil L\rceil 2^N} &\simeq \overline{\eta_N \Big|_{\left[0, \frac{k}{\lceil L\rceil 2^N}\right]}} * \eta_N \Big|_{\left[0, \frac{k+1}{\lceil L\rceil 2^N}\right]} \\
&\simeq \overline{\eta_{N+1} \Big|_{\left[0, \frac{k}{\lceil L\rceil 2^N}\right]}} * \eta_{N+1} \Big|_{\left[0, \frac{k+1}{\lceil L\rceil 2^N}\right]} \\
&\simeq \eta_{k/\lceil L\rceil 2^N}^{2k+1/\lceil L\rceil 2^{N+1}} * \eta_{2k+1/\lceil L\rceil 2^{N+1}}^{k+1/\lceil L\rceil 2^N}.
\end{aligned}$$

Moreover, we will argue that the homotopy $H_{k/\lceil L\rceil 2^N}^{k+1/\lceil L\rceil 2^N}$ witnessing the equivalence in (8.9) can be taken such that its image is contained in the same open ball that the

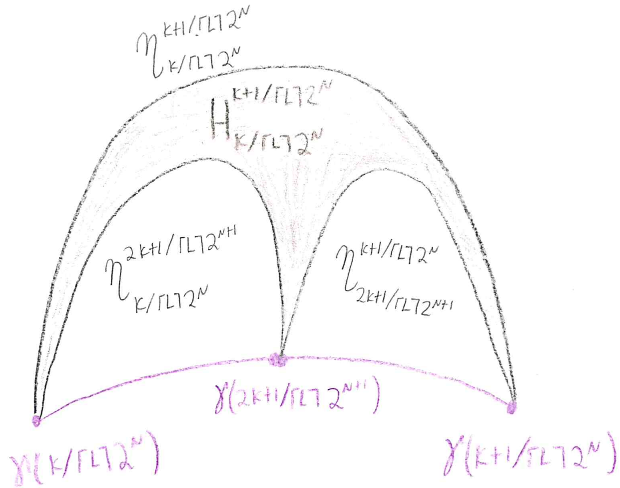

Figure 8.1: Relationship between γ, η_N, and H.

image of $\eta^{k+1/\lceil L\rceil 2^N}_{k/\lceil L\rceil 2^N}$ is contained in:

$$\operatorname{Im}\left(H^{k+1/\lceil L\rceil 2^N}_{k/\lceil L\rceil 2^N}\right) \subset B\left(\gamma\left(\frac{k}{\lceil L\rceil 2^N}\right), \frac{1}{2^{N-1}} + 2\varepsilon_N\right). \tag{8.10}$$

By the equivalence of paths stated in (8.9), there exists a Lipschitz map $H^{k+1/\lceil L\rceil 2^N}_{k/\lceil L\rceil 2^N} : \mathbb{D}^2 \to (M, d)$ whose image of the boundary of the 2-disk is the image of these paths:

$$\operatorname{Im}\left(H^{k+1/\lceil L\rceil 2^N}_{k/\lceil L\rceil 2^N}\Big|_{\partial\mathbb{D}^2}\right) = \operatorname{Im}\left(\eta^{k+1/\lceil L\rceil 2^N}_{k/\lceil L\rceil 2^N}\right) \cup \operatorname{Im}\left(\eta^{2k+1/\lceil L\rceil 2^{N+1}}_{k/\lceil L\rceil 2^N} * \eta^{k+1/\lceil L\rceil 2^N}_{2k+1/\lceil L\rceil 2^{N+1}}\right).$$

By Lemma 7.0.10, the Lipschitz map $H^{k+1/\lceil L\rceil 2^N}_{k/\lceil L\rceil 2^N}$ can be taken such that its image is contained in the image of the boundary of the 2-disk. Thus, the image of $H^{k+1/\lceil L\rceil 2^N}_{k/\lceil L\rceil 2^N}$ is contained in the images of these paths.

By construction, the images of these paths are contained the following open balls:

$$
\begin{aligned}
\mathrm{Im}\left(\eta_{k/\lceil L\rceil 2^N}^{k+1/\lceil L\rceil 2^N}\right) &\subset B\left(\gamma\left(\frac{k}{\lceil L\rceil 2^N}\right), \frac{1}{2^{N-1}} + 2\varepsilon_N\right), \\
\mathrm{Im}\left(\eta_{k/\lceil L\rceil 2^N}^{2k+1/\lceil L\rceil 2^{N+1}}\right) &\subset B\left(\gamma\left(\frac{k}{\lceil L\rceil 2^N}\right), \frac{1}{2^{N}} + 2\varepsilon_{N+1}\right), \\
\mathrm{Im}\left(\eta_{2k+1/\lceil L\rceil 2^{N+1}}^{k+1/\lceil L\rceil 2^N}\right) &\subset B\left(\gamma\left(\frac{2k+1}{\lceil L\rceil 2^{N+1}}\right), \frac{1}{2^{N}} + 2\varepsilon_{N+1}\right).
\end{aligned}
$$

It is immediate that the images $\mathrm{Im}\left(\eta_{k/\lceil L\rceil 2^N}^{k+1/\lceil L\rceil 2^N}\right)$ and $\mathrm{Im}\left(\eta_{k/\lceil L\rceil 2^N}^{2k+1/\lceil L\rceil 2^{N+1}}\right)$ are contained in the open ball $B\left(\gamma\left(\frac{k}{\lceil L\rceil 2^N}\right), \frac{1}{2^{N-1}} + 2\varepsilon_N\right)$. To see that any point in $\mathrm{Im}\left(\eta_{2k+1/\lceil L\rceil 2^{N+1}}^{k+1/\lceil L\rceil 2^N}\right)$ is also contained in this open ball, we will use the triangle inequality and that the path γ is Lipschitz.

First, we will make an observation about the Lipschitz constant L_γ associated to γ. Since Γ is a L-Lipschitz map and a lift of the path γ, and the map π is a metric map, we have the following string of inequalities for any $t, t' \in I$,

$$
\begin{aligned}
d(\gamma(t), \gamma(t')) &= d(\pi \circ \Gamma(t), \pi \circ \Gamma(t')) \\
&\leq d_{\mathcal{P}}(\Gamma(t), \Gamma(t')) \\
&\leq L \cdot |t - t'|.
\end{aligned}
$$

Thus, the Lipschitz constant L_γ associated to the map γ is no larger than L and we have the inequalities,

$$L_\gamma \leq L \leq \lceil L\rceil.$$

Now, since γ is Lipschitz and the just established inequalities, we have a bound

on the distance between $\gamma\left(\frac{k}{\lceil L\rceil 2^N}\right)$ and $\gamma\left(\frac{2k+1}{\lceil L\rceil 2^{N+1}}\right)$:

$$d\left(\gamma\left(\frac{k}{\lceil L\rceil 2^N}\right), \gamma\left(\frac{2k+1}{\lceil L\rceil 2^{N+1}}\right)\right) \leq \frac{L_\gamma}{\lceil L\rceil 2^{N+1}} \leq \frac{1}{2^{N+1}}.$$

Let $p \in \operatorname{Im}\left(\eta^{k+1/\lceil L\rceil 2^N}_{2k+1/\lceil L\rceil 2^{N+1}}\right)$. As noted above, the point p is also contained in the open ball $B\left(\gamma\left(\frac{2k+1}{\lceil L\rceil 2^{N+1}}\right), \frac{1}{2^N} + 2\varepsilon_{N+1}\right)$. Then, by THE triangle inequality, we have the following string of inequalities:

$$\begin{aligned} d\left(p, \gamma\left(\frac{k}{\lceil L\rceil 2^N}\right)\right) &\leq d\left(p, \gamma\left(\frac{2k+1}{\lceil L\rceil 2^{N+1}}\right)\right) + d\left(\gamma\left(\frac{2k+1}{\lceil L\rceil 2^{N+1}}\right), \gamma\left(\frac{k}{\lceil L\rceil 2^N}\right)\right) \\ &\leq \frac{1}{2^N} + 2\varepsilon_{N+1} + \frac{1}{2^{N+1}} \\ &< \frac{1}{2^{N-1}} + 2\varepsilon_N. \end{aligned}$$

Therefore the point p and the entire image $\operatorname{Im}\left(\eta^{k+1/\lceil L\rceil 2^N}_{2k+1/\lceil L\rceil 2^{N+1}}\right)$ is contained in the open ball $B\left(\gamma\left(\frac{k}{\lceil L\rceil 2^N}\right), \frac{1}{2^{N-1}} + 2\varepsilon_N\right)$. Therefore, a homotopy as in (8.10) exists.

We can reparametrize each homotopy $H^{k+1/\lceil L\rceil 2^N}_{k/\lceil L\rceil 2^N}$ such that the domain is the rectangle $\left[\frac{k}{\lceil L\rceil 2^N}, \frac{k+1}{\lceil L\rceil 2^N}\right] \times \left[\Sigma_{i=1}^N \frac{1}{\lceil L\rceil 2^i}, \Sigma_{i=1}^{N+1} \frac{1}{\lceil L\rceil 2^i}\right]$ where the following equalities follow from the map being a homotopy:

$$\begin{aligned} H^{k+1/\lceil L\rceil 2^N}_{k/\lceil L\rceil 2^N}\left(\frac{k}{\lceil L\rceil 2^N}, s\right) &= \gamma\left(\frac{k}{\lceil L\rceil 2^N}\right) \text{ for all } s \in \left[\Sigma_{i=1}^N \frac{1}{\lceil L\rceil 2^i}, \Sigma_{i=1}^{N+1} \frac{1}{\lceil L\rceil 2^i}\right], \\ H^{k+1/\lceil L\rceil 2^N}_{k/\lceil L\rceil 2^N}\left(\frac{k+1}{\lceil L\rceil 2^N}, s\right) &= \gamma\left(\frac{k+1}{\lceil L\rceil 2^N}\right) \text{ for all } s \in \left[\Sigma_{i=1}^N \frac{1}{\lceil L\rceil 2^i}, \Sigma_{i=1}^{N+1} \frac{1}{\lceil L\rceil 2^i}\right], \\ H^{k+1/\lceil L\rceil 2^N}_{k/\lceil L\rceil 2^N}\left(t, \Sigma_{i=1}^N \frac{1}{\lceil L\rceil 2^i}\right) &= \eta^{k+1/\lceil L\rceil 2^N}_{k/\lceil L\rceil 2^N}(t) \text{ for all } t \in \left[\frac{k}{\lceil L\rceil 2^N}, \frac{k+1}{\lceil L\rceil 2^N}\right], \\ H^{k+1/\lceil L\rceil 2^N}_{k/\lceil L\rceil 2^N}\left(t, \Sigma_{i=1}^{N+1} \frac{1}{\lceil L\rceil 2^i}\right) &= \eta^{2k+1/\lceil L\rceil 2^{N+1}}_{k/\lceil L\rceil 2^N} * \eta^{k+1/\lceil L\rceil 2^N}_{2k+1/\lceil L\rceil 2^{N+1}}(t) \text{ for all } t \in \left[\frac{k}{\lceil L\rceil 2^N}, \frac{k+1}{\lceil L\rceil 2^N}\right]. \end{aligned}$$

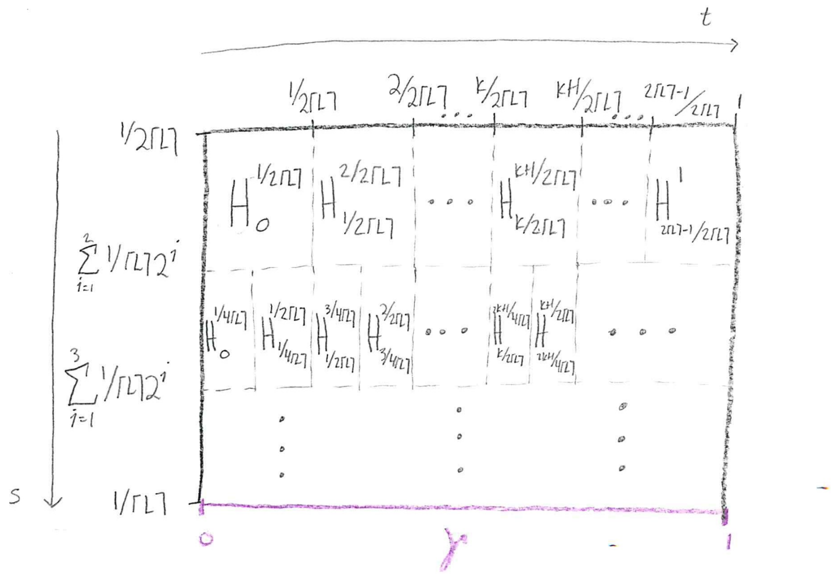

Figure 8.2: Domain of H.

We now construct a Lipschitz homotopy H from the path $\eta_1 \in \Gamma(1)$ to the path γ. The existence of such a Lipschitz homotopy will complete the proof.

Construct a map $H : I \times \left[\frac{1}{2\lceil L\rceil}, \frac{1}{\lceil L\rceil}\right] \to M$ by the following assignment

$$H(t,s) := \begin{cases} H^{k+1/\lceil L\rceil 2^N}_{k/\lceil L\rceil 2^N}(t,s) & , (t,s) \in \left[\frac{k}{\lceil L\rceil 2^N}, \frac{k+1}{\lceil L\rceil 2^N}\right] \times \left[\Sigma_{i=1}^{N} \frac{1}{\lceil L\rceil 2^i}, \Sigma_{i=1}^{N+1} \frac{1}{\lceil L\rceil 2^i}\right] \\ \gamma(t) & , s = 1/\lceil L\rceil. \end{cases}$$

See Figure 8.2 for a schematic for how the map H is defined on its domain. Since each homotopy $H^{k+1/\lceil L\rceil 2^N}_{k/\lceil L\rceil 2^N}$ is Lipschitz and H glues these maps compatibly, it is immediate that H is Lipschitz on $I \times \left[\frac{1}{2\lceil L\rceil}, \frac{1}{\lceil L\rceil}\right)$. We focus on showing that H is Lipschitz when $s = 1/\lceil L\rceil$.

We endow $I \times \left[\frac{1}{2\lceil L\rceil}, \frac{1}{\lceil L\rceil}\right]$ with the L^1 metric: for $(t,s), (t',s') \in I \times \left[\frac{1}{2\lceil L\rceil}, \frac{1}{\lceil L\rceil}\right]$,

$$d^1((t,s),(t',s')) = |t - t'| + |s - s'|.$$

Let $(t', 1), (t, s) \in I \times \left[\frac{1}{2\lceil L\rceil}, \frac{1}{\lceil L\rceil}\right]$. Since H is the path γ when $s = 1$ and γ is L_γ-Lipschitz, we have the following string of inequalities:

$$\begin{aligned} d(H(t',1), H(t,s)) &\leq d(H(t',1), H(t,1)) + d(H(t,1), H(t,s)) \\ &= d(\gamma(t'), \gamma(t)) + d(H(t,1), H(t,s)) \\ &\leq L_\gamma |t' - t| + d(H(t,1), H(t,s)). \end{aligned}$$

We now aim to show that H is Lipschitz on vertical slices in its domain. This will enable us to show that H is Lipschitz when $s = 1/\lceil L\rceil$.

Fix $t \in I$. Let (s_n) be a monotonic sequence in $\left[\frac{1}{2\lceil L\rceil}, \frac{1}{\lceil L\rceil}\right)$ that converges to $1/\lceil L\rceil$. Consider the sequence in $\mathbb{R}$

$$\frac{d\left(H(t, s_n), H\left(t, \frac{1}{\lceil L\rceil}\right)\right)}{d^1\left((t, s_n), \left(t, \frac{1}{\lceil L\rceil}\right)\right)} = \frac{d\left(H(t, s_n), \gamma(t)\right)}{\frac{1}{\lceil L\rceil} - s_n}. \tag{8.11}$$

It is enough to show that there is an upper bound for this sequence that does not depend on (s_n).

For each n, since the sequence (s_n) converges to $\frac{1}{\lceil L\rceil}$, there exists a positive integer $N(s_n)$ and an integer $k \in \{0, 1, \ldots, \lceil L\rceil 2^{N(s_n)} - 1\}$ such that

$$(t, s_n) \in \left[\frac{k}{\lceil L\rceil 2^{N(s_n)}}, \frac{k+1}{\lceil L\rceil 2^{N(s_n)}}\right] \times \left[\Sigma_{i=1}^{N(s_n)} \frac{1}{\lceil L\rceil 2^i}, \Sigma_{i=1}^{N(s_n)+1} \frac{1}{\lceil L\rceil 2^i}\right].$$

So, we have the following bounds on $\frac{1}{\lceil L\rceil} - s_n$:

$$\frac{1}{\lceil L \rceil 2^{N(s_n)+1}} \le \frac{1}{\lceil L \rceil} - s_n \le \frac{1}{\lceil L \rceil 2^{N(s_n)}}. \tag{8.12}$$

Also, by construction of H, we know that $H(t, s_n)$ is contained in an open ball centered at a point in $\mathrm{Im}(\gamma)$:

$$H(t, s_n) = H_{k/\lceil L \rceil 2^{N(s_n)}}^{k+1/\lceil L \rceil 2^{N(s_n)}}(t, s_n) \in B\left(\gamma\left(\frac{k}{\lceil L \rceil 2^{N(s_n)}}\right), \frac{1}{2^{N(s_n)-1}} + 2\varepsilon_{N(s_n)}\right). \tag{8.13}$$

Additionally, since γ is L_γ-Lipschitz and $t \in \left[\frac{k}{\lceil L \rceil 2^{N(s_n)}}, \frac{k+1}{\lceil L \rceil 2^{N(s_n)}}\right]$, we have the following inequality:

$$d\left(\gamma\left(\frac{k}{\lceil L \rceil 2^{N(s_n)}}\right), \gamma(t)\right) \le L_\gamma \left|\frac{k}{\lceil L \rceil 2^{N(s_n)}} - t\right| \le \frac{L_\gamma}{\lceil L \rceil 2^{N(s_n)}} \le \frac{1}{2^{N(s_n)}}. \tag{8.14}$$

From (8.13) and (8.14) and the triangle inequality, we have an upper bound for the numerator in (8.11):

$$\begin{aligned} d\left(H(t, s_n), \gamma(t)\right) &\le d\left(H(t, s_n), \gamma\left(\frac{k}{\lceil L \rceil 2^{N(s_n)}}\right)\right) + d\left(\gamma\left(\frac{k}{\lceil L \rceil 2^{N(s_n)}}\right), \gamma(t)\right) \\ &< \frac{1}{2^{N(s_n)-1}} + 2\varepsilon_{N(s_n)} + \frac{1}{2^{N(s_n)}} \\ &= \frac{3}{2^{N(s_n)}} + 2\varepsilon_{N(s_n)}. \end{aligned}$$

Finally, combining the inequalities in (8.15) and (8.12), we arrive at an upper bound for (8.11):

$$
\begin{aligned}
\frac{d\left(H(t,s_n),\gamma(t)\right)}{\frac{1}{\lceil L\rceil}-s_n} &< \left(\frac{3}{2^{N(s_n)}}+2\varepsilon_{N(s_n)}\right)\left(\lceil L\rceil 2^{N(s_n)+1}\right)\\
&= 8\lceil L\rceil.
\end{aligned}
$$

Thus, the sequence $\frac{d\left(H(t,s_n),H\left(t,\frac{1}{\lceil L\rceil}\right)\right)}{d^1\left((t,s_n),\left(t,\frac{1}{\lceil L\rceil}\right)\right)}$ has an upperbound for any monotonically increasing sequence (s_n) that converges to $1/\lceil L\rceil$.

Therefore, for any $t \in I$, the function $H(t,-) : \left[\frac{1}{2\lceil L\rceil},\frac{1}{\lceil L\rceil}\right] \to M$ is Lipschitz. Moreover, for any t the Lipschitz constant for $H(t,-)$ is bounded by $8\lceil L\rceil$. Continuing with the string of inequalities in (8.11), we have the following:

$$
\begin{aligned}
d(H(t',1),H(t,s)) &\leq L_\gamma|t'-t| + d(H(t,1),H(t,s))\\
&\leq L_\gamma|t'-t| + 8\lceil L\rceil|1-s|\\
&\leq \lceil L\rceil|t'-t| + 8\lceil L\rceil|1-s|\\
&\leq \lceil L\rceil d^1((t',1),(t,s)) + 8\lceil L\rceil d^1((t',1),(t,s))\\
&= 9\lceil L\rceil d^1((t',1),(t,s)).
\end{aligned}
$$

Thus, H is Lipschitz on its domain. So, the original path γ is homotopic to $\eta_1 \in \Gamma(1)$. Thus, we have equality of classes $\Gamma(1) = \widehat{\gamma}(1)$. Therefore, the lift $\Gamma = \widehat{\gamma}$ agrees with the standard lift.

□

Unique lifting property

We will now show that the universal path space of a purely 2-unrectifiable metric space, in particular of a contact 3-manifold, satisfies a unique lifting property as was

suggested in Claim 8.0.0.1

Definition 8.0.47. A metric space (X, d^X) is a *path space* if the metric is realized as the infimum of diameters of paths joining points, that is, for points $x, x' \in X$,

$$d^X(x, x') = \inf\{|\operatorname{Im}(\eta)| \;:\; \eta \text{ is a Lipschitz path joining } x \text{ and } x'\}.$$

Observation 8.0.48. As was argued in the proof of Lemma 8.0.29, a geodesic metric space, i.e., a metric space in which two points can be joined by a length-minimizing geodesic, is a path space. In particular, all complete sub-Riemannian manifolds are path spaces.

Remark 8.0.49. It is likely true that each connected component of a Riemannian manifold is a path space as all such spaces are locally geodesically convex.

Lemma 8.0.50. *Let (M, d) be a based, purely 2-unrectifiable metric space with base point $p_0 \in M$. Let (X, d^X) be a based, Lipschitz path-connected path space with base point $x_0 \in X$ and let $f : (X, d^X) \to (M, d)$ be a based, Lipschitz map such that the induced homomorphism on first Lipschitz homotopy groups*

$$f_\# : \pi_1^{Lip}((X, d^X), x_0) \longrightarrow \pi_1^{Lip}((M, d), p_0)$$

is constant at $[p_0]$. Then, there exists a unique, based, Lipschitz lift of f:

$$\begin{tikzcd} & \mathcal{P}_{(M,d)} \arrow[d, "\pi"] \\ X \arrow[ur, dashed, "\widehat{f}", "\exists!"'] \arrow[r, "f"] & M. \end{tikzcd}$$

Proof. We will begin by defining a lift of the map f and arguing that the lift is in fact unique. The argument will be the standard such argument. Then, we will argue

that the lift is Lipschitz which will follow from the map f being Lipschitz and the metric space (X, d^X) being a path space.

We will now define a map $\widehat{f} : X \to \mathcal{P}_{(M,d)}$ using that the metric space X is Lipschitz path-connected. Let $x \in X$. Since X is Lipschitz path-connected, there exists a Lipschitz path $\gamma : (I, 0, 1) \to (X, x_0, x)$ joining the base point x_0 to x. Thus, the composition $f \circ \gamma$ is a Lipschitz path in M joining the base point p_0 to the point $f(x)$. So, by Lemma 8.0.37, there is a Lipschitz lift $\widehat{f \circ \gamma}$ of the path. Define the map $\widehat{f}$ by the assignment:

$$\begin{aligned} \widehat{f} \quad X &\to \quad \mathcal{P}_{(M,d)} \\ x &\mapsto \widehat{f \circ \gamma}(1). \end{aligned}$$

Before showing that the map $\widehat{f}$ is well-defined, note that since $\widehat{f \circ \gamma}$ is a lift

$$\pi \circ f(x) = \pi \circ \widehat{f \circ \gamma}(1) = f \circ \gamma(1) = f(x).$$

Thus, provided that the map is well-defined, $\widehat{f}$ is a lift of the given map f.

To show that the map $\widehat{f}$ is well-defined, let $\gamma' : (I, 0, 1) \to (X, x_0, x)$ be another Lipschitz path joining x_0 and x. Then, the concatenation $\gamma' * \overline{\gamma}$ is a Lipschitz loop in X that is based at x_0 and post-composing by f yields a Lipschitz loop in M based at p_0:

$$f \circ \gamma' * \overline{\gamma} = f \circ \gamma' * \overline{f \circ \gamma}.$$

Since f induces the constant homomorphism between first Lipschitz homotopy groups, the loop $f \circ \gamma' * \overline{f \circ \gamma}$ is Lipschitz null-homotopic. Thus, the paths $f \circ \gamma'$ and $f \circ \gamma$ joining p_0 to $f(x)$ are Lipschitz homotopic. Thus, the map $\widehat{f}$ is well-defined as we have the following equality in $\mathcal{P}_{(M,d)}$:

$$\widehat{f \circ \gamma'}(1) = [f \circ \gamma'] = [f \circ \gamma] = \widehat{f \circ \gamma}(1).$$

Now we will show that $\widehat{f}$ is the unique lift of the maps f. Let $F : X \to \mathcal{P}_{(M,d)}$ be a lift of f and take a point $x \in X$. Since X is Lipschitz path-connected, there exists a Lipschitz path γ joining the base point x_0 to x. The composition $F \circ \gamma : I \to \mathcal{P}_{(M,d)}$ is then a lift of the path $f \circ \gamma$. By Theorem 8.0.46, since lifts of paths are unique, we have an equality of paths $F \circ \gamma = \widehat{f \circ \gamma}$. Thus, the map F agrees with the lift $\widehat{f}$ as

$$F(x) = F \circ \gamma(1) = \widehat{f \circ \gamma}(1) = \widehat{f}(x).$$

Finally, we will show that the lift $\widehat{f}$ is Lipschitz. Let L be the Lipschitz constant for the Lipschitz map f. Let $x, x' \in X$ and consider the value $d_{\mathcal{P}}(\widehat{f}(x), \widehat{f}(x'))$. Since the classes $\widehat{f}(x)$ and $\widehat{f}(x')$ do not depend on the paths joining the base point x_0 to x and x' respectively, we can rewrite the distance as

$$d_{\mathcal{P}}(\widehat{f}(x), \widehat{f}(x')) \quad = \quad \inf_{\gamma,\gamma'} d_{\mathcal{P}}([f \circ \gamma], [f \circ \gamma']),$$

where the infimum is taken over all Lipschitz paths γ joining x_0 to x and all Lipschitz paths γ' joining x_0 to x'.

Since f is L-Lipschitz, we have the following string of inequalities. In the following, α is a Lipschitz path in M joining $f(x')$ and $f(x)$ and η is a Lipschitz path in X joining x' and x.

$$\begin{aligned}
\inf_{\gamma,\gamma'} d_{\mathcal{P}}([f \circ \gamma], [f \circ \gamma']) &= \inf_{\alpha,\gamma,\gamma'} \{|\operatorname{Im}(\alpha)| \;:\; \alpha \simeq \overline{f \circ \gamma'} * f \circ \gamma = f \circ (\overline{\gamma}' * \gamma)\} \\
&\le \inf_{\eta,\gamma,\gamma'} \{|\operatorname{Im}(f \circ \eta)| \;:\; \eta \simeq \overline{\gamma}' * \gamma\} \\
&\le L \cdot \inf_{\eta,\gamma,\gamma'} \{|\operatorname{Im}(\eta)| \;:\; \eta \simeq \overline{\gamma}' * \gamma\}.
\end{aligned}$$

Since X is path-connected, any path η in X can be written as the concatenation $\overline{\gamma}' * \gamma$ of based paths γ and γ'. Thus, the γ and γ' can be dropped from the last term in this last string of inequalities. So, with this in mind, and since X is a path space, we have the following equality:

$$\begin{aligned}\inf_{\eta,\gamma,\gamma'}\{|\operatorname{Im}(\eta)| \ : \ \eta \simeq \overline{\gamma}' * \gamma\} &= \inf_{\eta}\{|\operatorname{Im}(\eta)| \ : \ \eta \text{ is a Lipschitz path joining } x \text{ and } x'\} \\ &= d^X(x,x').\end{aligned}$$

Stringing these inequalities together, we arrive at the desired inequality

$$d_{\mathcal{P}}(\widehat{f}(x),\widehat{f}(x')) \leq L \cdot d^X(x,x').$$

Therefore, $\widehat{f}$ is Lipschitz. ☐

Theorem 8.0.51. *Let (M,d) be a based, purely 2-unrectifiable space and let (X,d^X) be a based, Lipschitz path-connected, path space. Let $f : (X,d^X) \to (M,d)$ be a Lipschitz map. Then, there exists a unique lift $\widehat{f} : X \to \mathcal{P}_{(M,d)}$ of the map f if and only if the induced homomorphism on first Lipschitz homotopy groups*

$$f_{\#} : \pi_1^{Lip}((X,d^X),x_0) \longrightarrow \pi_1^{Lip}((M,d),p_0)$$

is constant at $[p_0]$.

Proof. Lemma 8.0.50 yields the backwards direction. For the forwards direction, first since we assume that the universal path space has the unique lifting property, by Lemma 8.0.41, $\mathcal{P}_{(M,d)}$ is Lipschitz simply-connected.

Now, the map f can be expressed as the composition $\pi \circ \widehat{f}$. Thus, the map induced on first Lipschitz homotopy groups can also be expressed as a composition:

$$f_{\#} = (\pi \circ \widehat{f})_{\#} = \pi_{\#} \circ \widehat{f}_{\#}.$$

Since $\mathcal{P}_{(M,d)}$ is Lipschitz simply-connected, the homomorphism $\widehat{f}_{\#}$ is constant. Thus, so is the homomorphism $f_{\#}$. □

Remark 8.0.52. At present, for a contact 3-manifold (M, d_{CC}^M), it seems unlikely that the universal path space $\mathcal{P}_{(M,d)}$ with the metric $d_{\mathcal{P}}$ is quasi-convex. Recall that $\mathcal{P}_{(M,d_{CC}^M)}$ is quasi-convex if there exists a constant $c > 0$ such that for any points $[\gamma], [\gamma'] \in \mathcal{P}_{(M,d_{CC}^M)}$, there exists a path α such that

$$l^{\mathcal{P}_{(M,d)}}(\alpha) \leq c \cdot d_{\mathcal{P}}([\gamma], [\gamma']).$$

To see why this is unlikely, let $c > 0$ be an arbitrary constant and take $[\gamma'] = [p_0]$ to be the base point in $\mathcal{P}_{(M,d_{CC}^M)}$. Find an injective path γ_c based at p_0 that maps into the open ball $B_{CC}^M(p_0, 1/2)$ which has length greater than c. Think of such a path as extremely curly but always staying within the ball. So, the diameter of the image of the path $|\mathrm{Im}(\gamma)|$ is less than 1, but the length of the path is c. We would then expect the length of any path α in $\mathcal{P}_{(M,d_{CC}^M)}$ joining $[p_0]$ and $[\gamma_c]$ to have length at least c. If this is the case, we have the inequality

$$l^{\mathcal{P}_{(M,d)}}(\alpha) \geq l^{CC}(\gamma_c) \geq c \geq c \cdot |\mathrm{Im}(\gamma_c)| \geq c \cdot d_{\mathcal{P}}([\gamma], [p_0])$$

which implies that $\mathcal{P}_{(M,d_{CC}^M)}$ is not quasi-convex for any constant c. A possible fruitful adjustment would be to replace diameter in the definition of $d_{\mathcal{P}}$ with length. Diameter was used in this discussion in order to agree with the work done in [4].

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